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POLYCOPIE DE COURS :

Advanced Fluid Mechanics

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Expertise			

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Foreword

This course handout of Fluid Mechanics, in accordance with the official program of the Ministry of Higher Education and Scientific Research. It is intended for students of the first year of the Master of Renewable Energy in Mechanical Engineering.

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Chapter 1: Fluid dynamics and transport equations

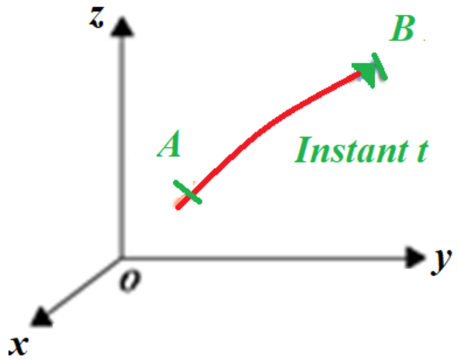
1.1 Description of the movement

1.1.1 Lagrange variable

the Lagrangian specification of the flow field is a way of looking at fluid motion where the observer follows an individual fluid parcel as it moves through space and time

A (a,b,c) the coordinates of a fluid particle at time t_o in the frame (0,x,y,z).

The independent coordinates (a, b, c, t) are called Lagrange variables



The coordinates of the particle at time t are M(x, y, z, t). The motion of the fluid is characterized by the following relations:

$$\begin{cases} x = f_1(a,b,c) \\ y = f_2(a,b,c) \\ z = f_3(a,b,c) \end{cases}$$

In this description of fluid motion, each particle is individually followed in its motion

1.1.2 Euler variable

To study fluid motion, it is often more convenient to use Euler variables. They allow, for example, to define the velocity field at each instant t and at any point M of the fluid.

In the O,x,y,z frame of reference the velocity vector has the following components:

$$\vec{V}(x,y,z) \equiv (u(x,y,z), v(x,y,z), w(x,y,z))$$

Euler's point of view is more convenient for the experimenter, because we place ourselves at a point M(x, y, z) of the fluid and we study the variations of physical quantities (for example the speed) at different times.

Euler's point of view is more convenient in kinematics because:

- for steady flows, the projection of the velocities in the frame does not depend on time

- the velocity vectors of the flow form a vector field to which the properties of vector fields can be applied

1.2 Streamline and stream tube

1.2.1 Definition of streamlines

A streamline is a tangent vectors to the instantaneous velocity direction (velocity is vector, and it has a magnitude and a direction) constitute the velocity vector field of the flow. These show the direction in which a massless fluid element will travel at any point in time**.**

1.2.2 Definition of stream tube

A stream tube is an imaginary tubular region of fluid surrounded by streamlines, which are lines drawn tangent to the instantaneous velocities of fluid particles. These streamlines form the walls of the stream tube and do not intersect except at points of zero velocity. For steady flow, the configuration of stream tubes remains fixed, while for unsteady flow, the configuration varies with time. Stream tubes are responsible for tangential fluid flow and prevent fluid from crossing the sides of the tube.

1.2.3 Streamline equation

Relative to an orthonormal frame, the differential equation of any streamline is written as:

$$\frac{dx}{u(x,y,z,t)} = \frac{dy}{v(x,y,z,t)} = \frac{dz}{w(x,y,z)}$$

u(x,y,z,t), v(x, y, z, t) and w(x, y, z, t) are the components of the speed in the frame (o, x, y,z)

In this formula, time is fixed

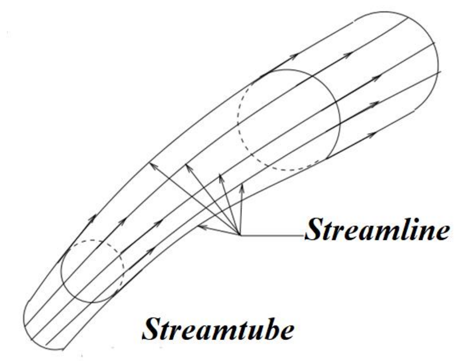


Fig 1.1 Streamline forming a stream tube

1.3 Pathlines

The pathlines are the trajectoiries that individual fluid particles follow. These can be thought of as "recording" the path of a fluid element in the flow over a certain period. The direction the path takes will be determined by the streamlines of the fluid at each moment in time.

The difference with the notion of streamline is that for the latter, we consider different particles at the same instant while the trajectory is relative to the same particle at different instants.

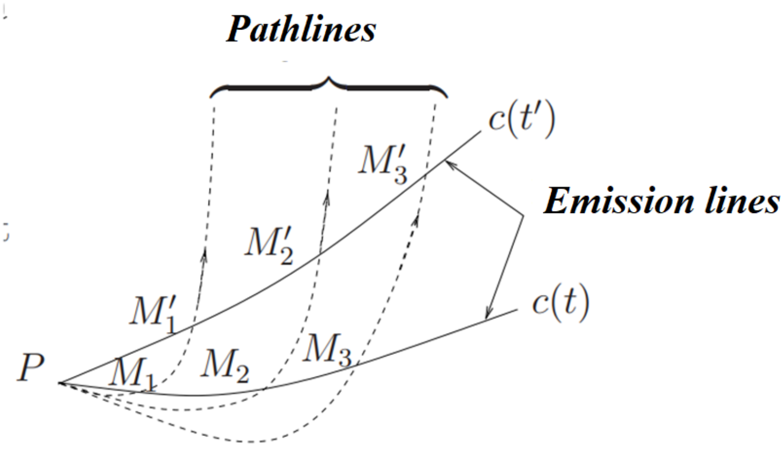


Fig 1.2 Pathline

The parametric differential equations of the trajectories are given by:

$$\begin{cases} \dfrac{dx}{dt} = u(x,y,z,t) \\ \dfrac{dy}{dt} = v(x,y,z,t) \\ \dfrac{dz}{dt} = w(x,y,z,t) \end{cases}$$

In these equations, time has become a variable

Definition

We consider a fluid particle be characterized by a scalar or vector quantity. The particle derivative of this quantity is the derivative with respect to time when we follow the particle in its motion.

Examples: velocity is the particle derivative of position and acceleration is the particle derivative of velocity.

We first consider this general case and seek to calculate the temporal evolution of a quantity f over a small volume around the point M. This speed of variation is called the total derivative of f and is noted df /dt.

If we denote by u_x the velocity of point M, the latter passes from position x at time t to position $x + u_x.dt$ at time t + dt. Consequently, the speed of variation is:

$$\frac{df}{dt} = \frac{f(x + dx, t + dt) - f(x, t)}{dt}$$

For dt sufficiently small, this speed is equal to:

$$\frac{df}{dt} = \frac{f(x + dx, t + dt) - f(x, t + dt)}{dt} + \frac{f(x, t + dt) - f(x, t)}{dt}$$

The second term on the righ is equal to the partial derivative $\frac{\partial f}{\partial t}$. Concerning the first, we can write the limited development:

$$f(x + dx, t + dt) = f(x, t + dt) + \frac{\partial f}{\partial x} dx$$

Generalizing to three dimensions, this allows us to write:

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial t} + (\vec{u}.\nabla)f}$$

1.4 Flow concepts

1.4.1 Definition of permanent (or stationary flow)

The permanent or stready flow is a flow in which the velocity of the fluid at a particular fixed point does not change with time

1.4.2 Conservative flow

Local equation and steady state

Steady state$\boldsymbol{\frac{\partial \rho}{\partial t} = 0}$ so $\boldsymbol{div\left(\rho \vec{V}\right) = 0}$

The resulting equation indicates that the flux $\boldsymbol{\rho v}$ of through a closed surface is zero

$$\iiint_\tau \boldsymbol{div\left(\rho \vec{V}\right) d\tau} = \iint_S \boldsymbol{\rho \vec{V}\, d\vec{S} = 0}$$

This equation therefore means the conservation of mass flow rate.

Case of incompressible fluid

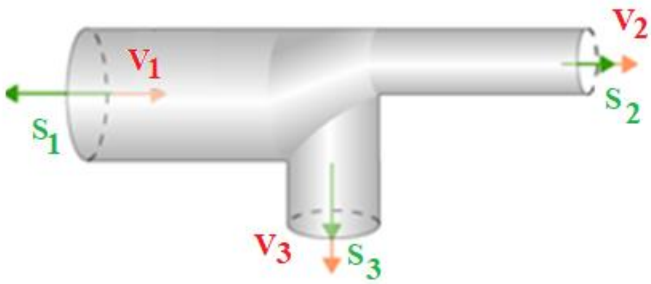
$\boldsymbol{\frac{\partial \rho}{\partial t} = 0}$ and $\boldsymbol{\rho = cst}$

$$\boldsymbol{div\,\vec{V} = 0}$$

So the flux of velocity through a closed surface is zero

$$\iiint_{\tau} \boldsymbol{div\,\vec{V}\,d\tau} = \iint_{S} \boldsymbol{\vec{V}\,d\vec{S} = 0}$$

The equation represents the conservation of flow rate in volume for an incompressible fluid



1.4.3 Conservation of mass equation with mass production in flow

Simply add the mass production term to the balance equation to obtain the local equation:

$$\boldsymbol{div}(\boldsymbol{\rho\vec{V}}) + \frac{\partial\rho}{\partial t} = \rho\dot{q}$$

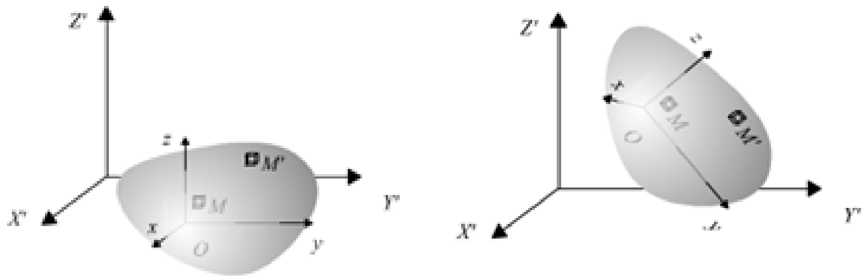
In this equation, q' represents mass production flow rate (in s^{-1})

q' > 0, represents a source and q'< 0 a well

Velocity field

Case of the ideal solid (non-deformable)

$$\overrightarrow{\acute{V}_M} = \overrightarrow{V_M} + \vec{\omega} \wedge \overrightarrow{MM'}$$



$\vec{\omega}$ is the rotation vector and:

$$\overrightarrow{V_M} = \vec{\omega} \wedge \overrightarrow{OM}$$

By calculating $\overrightarrow{rot}\,\overrightarrow{V_M}$, we can write $\overrightarrow{\acute{V}_M}$ in the form

$$\overrightarrow{\acute{V}_M} = \overrightarrow{V_M} + {}^{1}/_{2}\,\overrightarrow{rot}\,\overrightarrow{V_M} \wedge \overrightarrow{MM'}$$

Physical interpretation

The first term of the velocity expression represents a translation and the second a rotation of the solid.

The term $1/_2\,\overrightarrow{rot}\,\overrightarrow{V_M} = \vec{\omega}$ where $\vec{\omega}$ is the rotation vector.

1.5 Vorticity vector and strain rate tensor

Let us consider a volume element dt and two infinitely neighboring points M and M'. In the frame O, x, y, z the coordinates of M and M' are:

M(x, y, z) and M'(x+dx, y+dy, z+dz)

1.5.1 Expression of velocity

The coordinates of the velocity are:

$$\overrightarrow{V_M}(u(x,y,z), v(x,y,z), w(x,y,z))$$

$$\overrightarrow{V_M}\begin{pmatrix} u(x+dx, y+dy, z+dz) \\ v(x+dx, y+dy, z+dz) \\ w(x+dx, y+dy, z+dz) \end{pmatrix}$$

The coordinates of the speed in M' using the formula for finite increments.

For example, for the component along x:

$$u(x+dx, y+dy, z+dz, t) = u(x,y,z) + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$v_i(M') = v_i(M) + \sum_{j=1}^{3} \frac{\partial v_i}{x_j} dx_j$$

v_i represents a component of the velocity at point M'

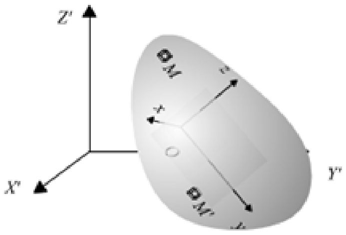
In vector form, we can write: $\overrightarrow{V'_M} = \overrightarrow{V_M} + \left(\overrightarrow{MM'}.\vec{\nabla}\right)\overrightarrow{V_M}$

To compare the previous expression with that of the perfect fluid and to highlight the meaning of the different terms, let us develop and calculate:

$$\overrightarrow{grad}\left(\vec{V}.\overrightarrow{MM'}\right) = \left(\overrightarrow{MM'}.\vec{\nabla}\right)\vec{V} + \left(\vec{V}.\vec{\nabla}\right)\overrightarrow{MM'} + \overrightarrow{MM'} \wedge \overrightarrow{rot}\,\vec{V} + \vec{V} \wedge \overrightarrow{rot}\,\overrightarrow{MM'}$$

We therefore obtain:

1.5.2 .Strain rate tensor



$$\overline{\overline{D}} = \frac{1}{2}\begin{bmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} & \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}+\frac{\partial w}{\partial z} \end{bmatrix} \text{Symetric tenser}$$

To highlight the physical meaning of the different terms, let's write the velocity in the form:

$$\overrightarrow{V'_M} = \overrightarrow{V_M} + \vec{\omega} \wedge \overrightarrow{MM'} + \overline{\overline{D}}\,\overrightarrow{MM'}$$

Note: si $\overline{\overline{D}} = 0$, the deformation rate is zero and we return to the case of the perfect non-deformable solid (or to the case of a deformable medium in relative absolute equilibrium)

Physical meaning of the different terms

$\overrightarrow{V_M}$: Represents an overall translation of the volume element

$\overline{\overline{D}}\,\overrightarrow{MM'}$: represents the deformation of the volume element

$\vec{\omega} \wedge \overrightarrow{MM'}$: is the moment of the vector $1/_2\,\overrightarrow{rot}\,\overrightarrow{V_M}$, it is the distribution of speeds during a block rotation of the volume element around an axis passing through M.

1.6 Acceleration

Expression of acceleration

$\vec{a}$ is the acceleration vector, by definition, we have: $\overrightarrow{a_{M,t}} = lim_{\Delta t \to 0} \frac{\overrightarrow{V_{M+dM,t+\Delta t}} - \overrightarrow{V_{M,t}}}{\Delta t}$

In the considered reference frame, $\overrightarrow{V_M}$ has the coordinates:

$$\overrightarrow{V_M}(u(x,y,z,t), v(x,y,z,t), w(x,y,z,t))$$

the coordinate of the speed along the x axis and calculates its differential:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial t}dt$$

This expression allows us to calculate the component of the acceleration on the x axis:

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dx}{dt} + \frac{\partial u}{\partial z}\frac{dx}{dt} + \frac{\partial u}{\partial t}$$

Vector form

We can write the previous expression:

$$a_x = \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{\partial u}{\partial t}$$

We can write the same type of relation for the components along y and z, so we obtain a vector relation:

$$\vec{a} = \frac{\overrightarrow{\partial V}}{\partial t} + (\vec{V}\vec{\nabla})\vec{V}$$

Other writing

Using a vector equality, we obtain:

$$\vec{a} = \frac{\overrightarrow{\partial V}}{\partial t} + {}^{1}/_{2}\,\overrightarrow{grad}\overrightarrow{V^2} + \overrightarrow{rot}\vec{V} \wedge \vec{V}$$

Physical interpretation:

$\frac{\overrightarrow{\partial V}}{\partial t}$ is called **local acceleration**, this term reflects the non-permanence of the flow, it is zero for a permanent flow

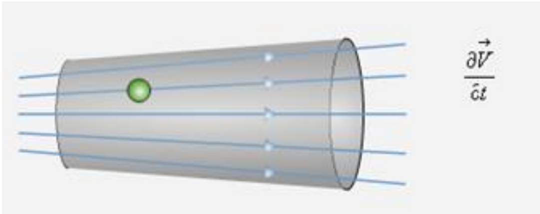
Convective acceleration:

$$(\vec{V}\vec{\nabla})\vec{V} = {}^{1}/_{2}\,\overrightarrow{grad}\overrightarrow{V^2} + \overrightarrow{rot}\overrightarrow{V} \wedge \vec{V}$$

Is the **convective acceleration**, this term reflects the non-uniformity of the flow.

To check whether a flow is steady, we place ourselves at a fixed point in the flow and measure the speed at different times.

To see if a flow is uniform, we measure the velocity at different points in the flow, at the same time.



1.7 Irrotational flow

Definition: Irrotational flow

Irrotational flow is a flow for which we have:

$$\overrightarrow{rot}\vec{V} = 0$$

From this equation, we deduce:

$$\vec{V} = \overrightarrow{grad}\Phi$$

An irrotational flow is a flow with potential velocities and vice versa

Continuity equation

$$div(\rho\vec{V}) + \frac{\partial\rho}{\partial t} = 0$$

For an incompressible fluid and an irrotational flow, we obtain:

$div\vec{V} = 0$ either $div(\overrightarrow{grad}\Phi) = 0$

So: $\boxed{\Delta\Phi = 0}$

Expression of acceleration

$$\vec{a} = \frac{\overrightarrow{\partial V}}{\partial t} + {}^{1}/_{2}\,\overrightarrow{grad}\overrightarrow{V^2}$$

Le terme$\overrightarrow{rot}\overrightarrow{V} \wedge \vec{V}$ est nul

Definition: Rotational flow and vorticity vector

The vorticity vector represents the instantaneous rotation speed vector:

$\vec{\omega} = {}^{1}/_{2}\,\overrightarrow{rot}\,\vec{V}$

Vortex line

A vortex line is a line tangent at each of its points to the vortex vector, it is such that:

$$\boxed{\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}}$$

Properties

By definition, the vortex vector field has a conservative flow:$\vec{\omega} = {}^1/_2\,\overrightarrow{rot}\,\vec{V} \Rightarrow div\vec{\omega} = 0$

$$\Rightarrow \oiint \vec{\omega}\overrightarrow{dS} = 0$$

Consequence: the flow of the vortex vector is constant in a vortex tube.

The intensity of the vortex tube is called the quantity:

$$I = \oiint \vec{\omega}\overrightarrow{dS}$$

$$\oiint \overrightarrow{rot}\,\vec{V}\,\overrightarrow{dS} = \oint \vec{V}\ \overrightarrow{dl} = \oiint \overrightarrow{2\omega dS} = 2I$$

1.8 Flow at velocities potential

This section is concerned with an important class of flow problems in which the vorticity is everywhere zero, and for such problems the Navier-Stokes equation may be greatly simplified. Finally, it may be shown that, when (∇ × V) is zero, one may describe the velocity by means of a scalar potential ϕ, using the equation

$$\vec{V} = \overrightarrow{grad}\Phi$$

Hypotheses: we assume the case of a perfect incompressible fluid in irrotational and permanent plane flow.

Dans ce cas, la vitesse dérive du potentiel Φ

$$\vec{V} = \overrightarrow{grad}\Phi$$

And the continuity equation is written:

$$\Delta\Phi = 0$$

To know the flow (streamline, velocity), it is therefore necessary to solve the Laplace equation. For two-dimensional flows, the method of complex potentials, described below, is very successful.

1.7.1 Conditions de Cauchy - Riemann

f(z) the function of the complex variable $x + iy$x + i y, $f(z)$ is differentiable on a domain D if, in the complex plane:

$\lim\limits_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ tends towards a finite limit $\frac{df(z)}{dz}$

We can put f(z) in the form :

$$f(z) = \Phi + i\Psi$$

Where Φ is the potential function and Ψ the stream function

1.7.2 Calculation of the derivative

$$\frac{df(z)}{dz} = \frac{d\Phi + id\Psi}{dx + idy}$$

For f f(Φ and Ψsatisfy the Laplace equation), this derivative must be independent of dz, that is, by developing:

$$\frac{df(z)}{dz} = \frac{\left(\frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy\right) + i\left(\frac{\partial\Psi}{\partial x}dx + \frac{\partial\Psi}{\partial y}dy\right)}{dx + idy}$$

$$\frac{df(z)}{dz} = \frac{\left(\frac{\partial\Phi}{\partial x} + i\frac{\partial\Psi}{\partial x}\right)dx + \left(\frac{\partial\Psi}{\partial y} + \frac{1}{i}\frac{\partial\Phi}{\partial y}\right)idy}{dx + idy}$$

$$\boxed{\frac{\partial\Phi}{\partial x} + i\frac{\partial\Psi}{\partial x} = \frac{\partial\Psi}{\partial y} + \frac{1}{i}\frac{\partial\Phi}{\partial y}}$$

1.7.3 Holomorphic function

The independence condition implies that:

$\frac{\partial\Phi}{\partial x} + i\frac{\partial\Psi}{\partial x} = \frac{\partial\Psi}{\partial y} + \frac{1}{i}\frac{\partial\Phi}{\partial y}$

We therefore obtain the relations $\begin{cases} \frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial x} = u \\ \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial y} = v \end{cases}$

u and *v* are the componantes of velocity. These are the Cauchy Riemann conditions***: the function f(z) is holomorphic on the domain D***

1.7.4 Uniform Stream in the x Direction

A uniform stream V = iU, possesses both a stream function and a velocity potential, which may be found as follows:

$\boldsymbol{\frac{\partial\Phi}{\partial x}} = \boldsymbol{\frac{\partial\Psi}{\partial x}} = \boldsymbol{u} = U \quad \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial y} = v = 0$

We may integrate each expression and discard the constants of integration, which do not affect the velocities in the flow. The results are :

A uniform stream $\Psi = Uy$ $\Phi = Ux$

The streamlines are horizontal straight lines (y = const), and the potential lines are vertical (x = const), that is, orthogonal to the streamlines, as expected as presented in the figure 1.5.a.

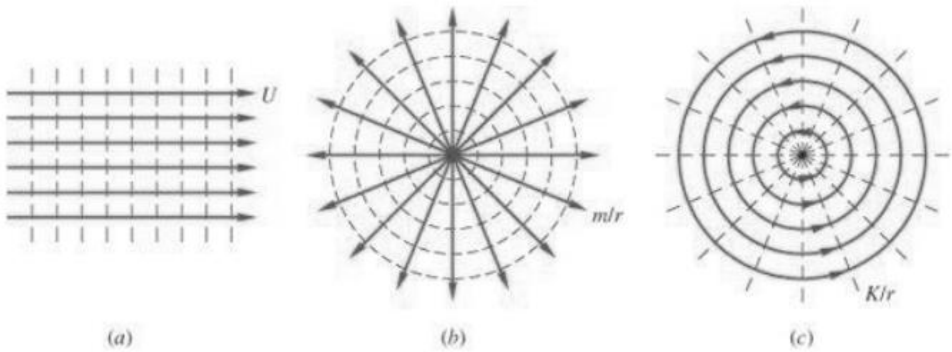


Fig 1.5 a uniform stream in the x direction (Solid lines are streamlines; dashed lines are potential lines)

1.7.5 Line Source or Sink at the Origin

Suppose that the z axis were a sort of thin pipe manifold through which fluid issued at total rate Q uniformly along its length b. Looking at the (xy) plane, we would see a cylindrical radial outflow or line source, as sketched in Figure 1.5b.

$$v_r = \frac{Q}{2\pi\, rb} = \frac{m}{r} = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \frac{\partial \Phi}{\partial r}$$

$$v_\theta = 0 = -\frac{\partial \Psi}{\partial r} = \frac{1}{r}\frac{\partial \Phi}{\partial \theta}$$

Where we have used the polar coordinate forms of the stream function and the velocity potential. Integrating and again discarding the constants of integration, we obtain the proper functions for this simple radial flow:

Line source or Sink:

$$\Psi = \mathrm{m}\theta \qquad \Phi = \mathrm{m}\ \ln \mathrm{r}$$

Where m = Q/(2πb) is a constant, positive for a source, negative for a sink. As shown in Figure (1.5b), the streamlines are radial spokes (constant θ), and the potential lines are circles (constant r)

1.7.6 Line Irrotational Vortex

A (two-dimensional) line vortex is a purely circulating steady motion, $\Psi_\theta = f(r)$ψθ only, $\Psi_\theta = 0$. This satisfies the continuity equation identically. We may also note that a variety of velocity distributions $\Psi_\theta(r)$ satisfy the θ momentum equation of a viscous fluid. The function $\Psi_\theta(r)$ is irrotational; that is, curl V = 0, and $\Psi_\theta(r) = k/r$, where K is a constant. This is sometimes called a free vortex, for which the stream function and velocity may be found:

$$v_r = 0 = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = \frac{\partial\Phi}{\partial r}$$

$$v_\theta = \frac{k}{r} = -\frac{\partial\Psi}{\partial r} = \frac{1}{r}\frac{\partial\Phi}{\partial\theta}$$

We may again integrate to determine the appropriate functions:

$$\Psi = -k\ln r \quad \text{and} \quad \Phi = K\,\theta$$

where K is a constant called the strength of the vortex. As shown in Figure. 1.3c, the streamlines are circles (constant r), and the potential lines are radial spokes (constant θ).

Note the similarity between Eqs. (8.13) and (8.14). A free vortex is a sort of reversed image of a source. The "bathtub vortex," formed when water drains through a bottom hole in a tank, is a good approximation to the free-vortex pattern. Each of the three elementary flow patterns in Fig. 8.3 is an incompressible irrotational flow and therefore satisfies both plane "potential flow" equations = 2χ = 0 and = 2 ϕ = 0. Since these are linear partial differential equations, any sum of such basic solutions is also a solution. Some of these composite solutions are quite interesting and useful

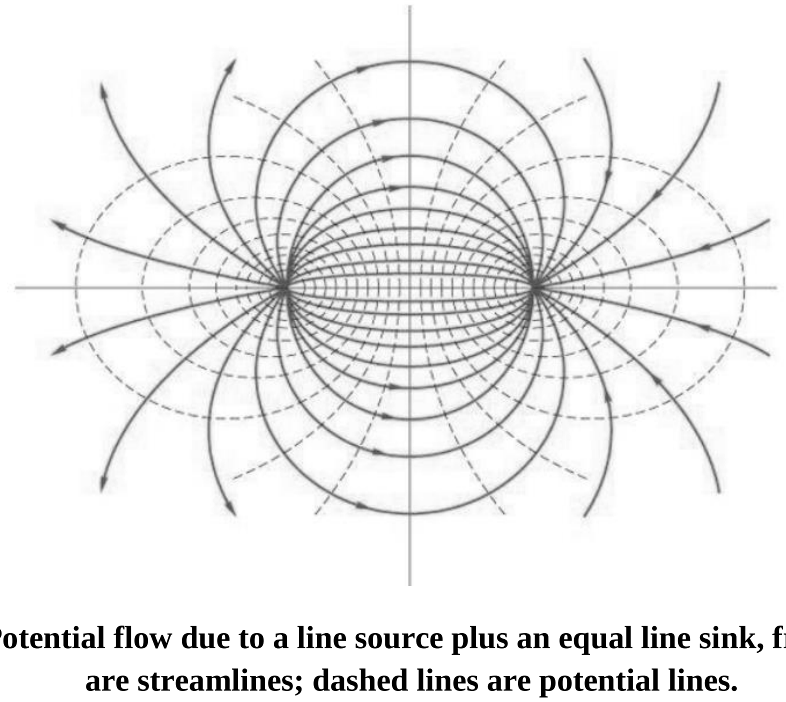


Figure 1.6. Potential flow due to a line source plus an equal line sink, from. Solid lines are streamlines; dashed lines are potential lines.

For example, consider a source +m at (x, y) = (−a, 0), combined with a sink of equal strength −m, placed at (+a, 0), as in Fig. 8.4. The resulting stream function is simply the sum of the two. In cartesian coordinates :

$$\Psi = \Psi_{Source} + \Psi_{Sink} = m\,tan^{-1}\frac{y}{x+a} - m\,tan^{-1}\frac{y}{x-a}$$

Similarly, the composite velocity potential is:

$$\Phi = \Phi_{Source} + \Phi_{Sink} = \frac{1}{2}m\ln[(x+a)^2 + y^2] - \frac{1}{2}m\ln[(x-a)^2 + y^2]$$

By using trigonometric and logarithmic identities, these may be simplified to:

Source plus Sink:

$$\boxed{\Psi = -m\,tan^{-1}\frac{2\,a\,y}{x^2 + y^2 - a^2}}$$

$$\boxed{\Phi = \frac{1}{2}m\ln\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}}$$

These lines are plotted in Figure (1.7) and are seen to be two families of orthogonal circles, with the streamlines passing through the source and sink and the potential lines encircling them. They are harmonic (laplacian) functions that are exactly analogous in electromagnetic theory to the electric current and electric potential patterns of a magnet with poles at (±a, 0).

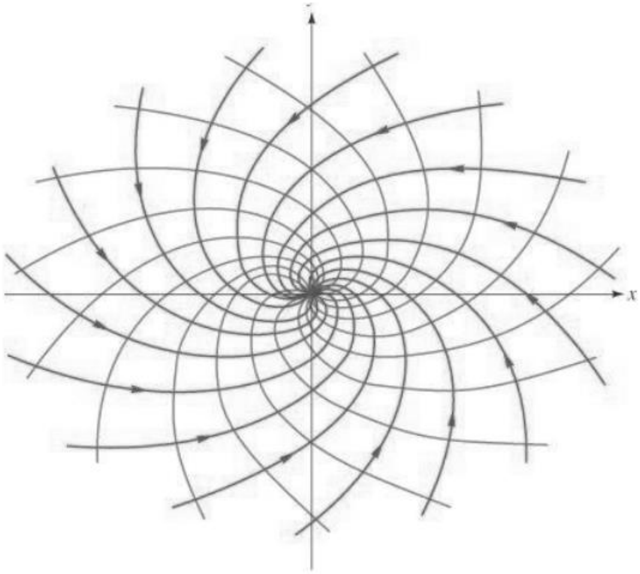


Fig. 1.7 Superposition of a sink plus a vortex, Eq. (8.16), simulates a tornado

Uniform Stream Plus a Sink at the Origin: The Rankine Half-Body

If we superimpose a uniform x-directed stream against an isolated source, a half-body shape appears. If the source is at the origin, the combined stream function is, in polar coordinates, We can set this equal to various constants and plot the streamlines, as shown in Figure (1.8). A curved, roughly elliptical, half-body shape appears, which separates the source flow from the stream flow. The body shape, which is named after the Scottish engineer W. J. M.

Rankine (1820–1872), is formed by the particular streamlines χ = ±πm. The half-width of the body far downstream is πm/U. The upper surface may be plotted from the relation:

$$r = \frac{m(\pi - \theta)}{U \sin\theta}$$

It is not a true ellipse. The nose of the body, which is a "stagnation" point where V = 0, stands at (x, y) = (−a, 0), where a = m/U. The streamline χ = 0 also crosses this point—recall that streamlines can cross only at a stagnation point.

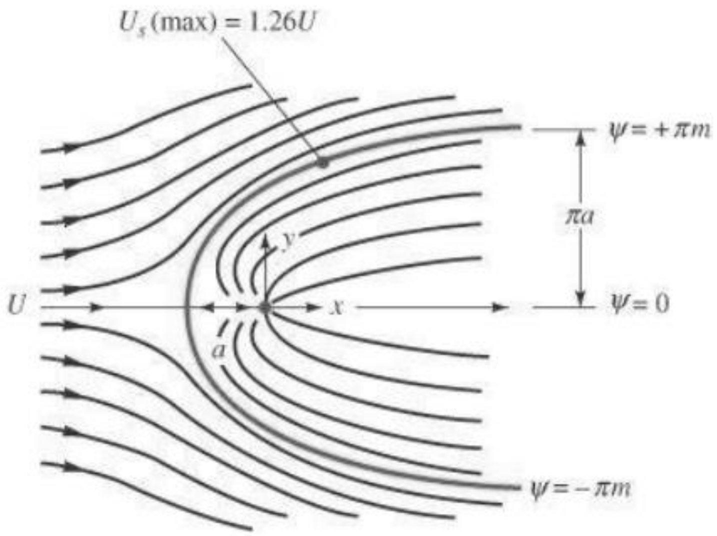


Fig. 1.8 Superposition of a source plus a uniform stream forms a Rankine half-body.

The cartesian velocity components are found by differentiation:

$$u = \frac{\partial \Psi}{\partial y} = U + \frac{m}{r} cos\theta \qquad\qquad v = -\frac{\partial \Psi}{\partial y} = \frac{m}{r}\sin\theta$$

Setting u = ψ = 0, we find a single stagnation point at θ = 180° and r = m/U, or (x, y) = (−m/U, 0), as stated. The resultant velocity at any point is,

$$U^2 = u^2 + v^2 = U^2\left(1 + \frac{a^2}{r^2} + \frac{2a}{r}\cos\theta\right)$$

where we have substituted m = Ua. If we evaluate the velocities along the upper surface χ = πm, we find a maximum value Us, max ≈ 1.26U at θ = 63°. This point is labeled in Figure(1.8) and, by Bernoulli’s equation, is the point of minimum pressure on the body surface. After this point, the surface flow decelerates, the pressure rises, and the viscous layer grows thicker and more susceptible to “flow separation”.

Chapter 2: Perfect Fluid and its Applications

2. Euler's equation of motion

In this chapter, we only consider fluids whose viscosity can be neglected; there is no friction between the different layers of fluids; these fluids are said to be perfect.

a. Euler equation
b. Local equation
c. General form

On each fluid volume element, we define:

ρ : the density

$\overrightarrow{F}$: Volume density of force (N/m^3).

$\vec{a}$: l'accélération par rapport au référentiel galiléen O, x, y, z.

Writing the integral equation

By writing the fundamental relation of dynamics relative to the Galilean frame of reference O,x ,y,z :

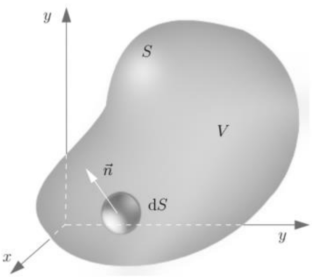


Fig 2.1 Control volume

$$\iiint_\tau \overrightarrow{F}\, d\tau + \iint_S -Pd\vec{S} = \iiint_\tau \rho\, \overrightarrow{a}\, d\tau$$

Using the gradient formula:

$$\iint_S -Pd\vec{S} = \iiint_\tau -\overrightarrow{grad}P\, d\tau$$

We get:

$$\iiint_\tau \left(\vec{F} - \overrightarrow{grad}P - \rho\, \overrightarrow{a}\right) d\tau = 0$$

The previous formula is true whatever the volume element chosen, we can therefore write:

$$\boxed{\vec{F} - \overrightarrow{grad}P - \rho\, \overrightarrow{a} = 0}$$

This equation represents the local form of Euler's equation (true at every point of the fluid).

From the balance of the forces applied to the fluid and the kinematic characteristics of the flow, it is this equation which will be used for the study of flows.Autres expressions de l'équation d'Euler

In the previous equation, the acceleration of the fluid is written (fluid kinematics):

$$\boxed{\vec{a} = \frac{\overrightarrow{dV}}{dt} = \frac{\overrightarrow{\partial V}}{\partial t} + {}^1\!/_2\, \overrightarrow{grad}\overrightarrow{V^2} + \overrightarrow{rot}\vec{V} \wedge \vec{V}}$$

<u>In this expression:</u>

$\frac{\overrightarrow{\partial V}}{\partial t}$ is the local acceleration (non-permanence of the flow).

${}^1\!/_2\, \overrightarrow{grad}\overrightarrow{V^2} + \overrightarrow{rot}\vec{V} \wedge \vec{V}$ is the convective acceleration (non-uniformity of the flow).

Remarks

A dynamic equation is insufficient for a complete study of a flow.

The characteristics of the flow of a fluid are given by:

- the speed V

- the pressure P

- the density ρ

- the temperature T

Euler's equation must therefore be supplemented by other equations characterizing the fluid, its movement and the flow conditions.

Elements to add

We must therefore add:

a. the mass conservation equation
b. $\boldsymbol{div}(\boldsymbol{\rho}\vec{\boldsymbol{V}}) + \frac{\partial \rho}{\partial t} = 0$
c. the equation of state of the fluid : $f(P, \rho, T) = 0$
d. the equation characterizing the type of transformation undergone by the fluid (incompressible, isothermal, adiabatic, etc.).
e. the boundary conditions and initial conditions which allow the determination of the integration constants.

2.2 Characteristic equation of the fluid

- incompressible liquid: $\rho = f(T)$

- slightly compressible liquid:$\rho = \rho_0(T)(1 + kP)$

- Perfect gas: $\frac{P}{\rho} = rT$

The transformations undergone

In the case of reversible transformations:

For isotherms: $\rho = cst$ (**incompressible fluid**) et $\frac{P}{\rho} = cst$ (**perfect gas**).

For adiabatic transformations: $\rho = cst$ (**incompressible fluid**) et $\frac{P}{\rho^\gamma} = cst$ (**perfect gas**).

Dynamic equation relative to the perfect fluid

Bernoulli relation Calculation hypotheses

Considering

- A perfect fluid (without viscosity)

- Incompressible (ρ = cste)

- In steady flow (partial derivatives with respect to time are zero)

- The volume force density derives from a potential U $\vec{F} = -\overrightarrow{grad}U$

- the walls limiting the fluid are fixed (no work provided) and adiabatic (no heat exchange with the exterior).**Première approche : équation dynamique**

In this first approach, we start from the dynamic equation (Euler's equation) and take into account the hypotheses:$\vec{F} - \overrightarrow{grad}P = \rho\,\vec{a}$

$$\vec{F} - \overrightarrow{grad}P = \rho\frac{\overrightarrow{\partial V}}{\partial t} + \rho\overrightarrow{grad}\frac{\overrightarrow{V^2}}{2} + \rho\overrightarrow{rot}\vec{V} \wedge \vec{V}$$

Taking up the hypotheses:

$\vec{F} = \overrightarrow{grad}U$ → Volume forces deriving from a potential

$\frac{\overrightarrow{\partial V}}{\partial t} = 0$ → the movement is permanent

$\rho\overrightarrow{grad}\frac{\overrightarrow{V^2}}{2} = \overrightarrow{grad}\left(\frac{\rho\,\overrightarrow{V^2}}{2}\right)$ → the fluid is incompressible

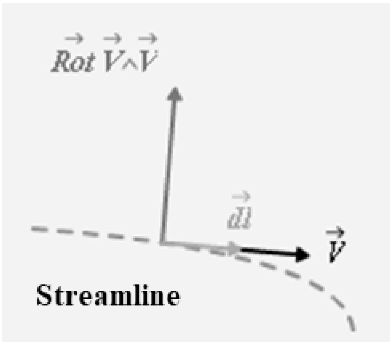
Demonstration

We obtain by replacing in the dynamic equation:

$$-\overrightarrow{grad}\left(P + U + \frac{\rho\,\overrightarrow{V^2}}{2}\right) = \rho\,\overrightarrow{rot}\,\vec{V} \wedge \vec{V}$$

Considering a streamline and take the elementary circulation of the two preceding terms:$-\overrightarrow{grad}\left(P + U + \frac{\rho\,\overrightarrow{V^2}}{2}\right).\overrightarrow{dl} = \rho\,\overrightarrow{rot}\vec{V} \wedge \vec{V}.\overrightarrow{dl}$

We note that the second term is zero ($\vec{V}$ *and* $\overrightarrow{dl}$ are collinear).



<u>Result</u>

We therefore obtain:

$\overrightarrow{grad}\left(P + U + \frac{\rho\,V^2}{2}\right).\overrightarrow{dl} = 0$

Using the gradient theorem, we get:

$d\left(P + U + \frac{\rho\,V^2}{2}\right) = 0 \Longrightarrow P + U + \frac{\rho\,V^2}{2} = cst$

Analysis of the result shows that the unit is the pascal (i.e. the joule per cubic meter). The sum of these three terms therefore represents the mechanical energy of the fluid per unit volume and this is constant along a streamline.

In general, the volume forces are the gravitational forces and the potential U is written as U = ρgz with g the acceleration of gravity and z the position of the fluid particle considered.

Conclusion

Bernoulli's Principle Formula is therefore written:

$$\boxed{P + \rho g z + \frac{\rho V^2}{2} = cst}$$

This equation is applies, in the case of a perfect, incompressible fluid in permanent motion, in the case where the volume forces are the forces of gravity with fixed walls and without heat exchange with the exterior.

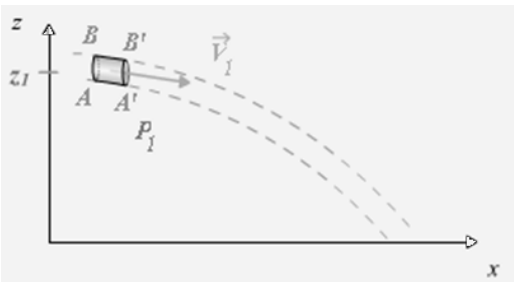
Physical meaning: it is an equation of conservation of energy

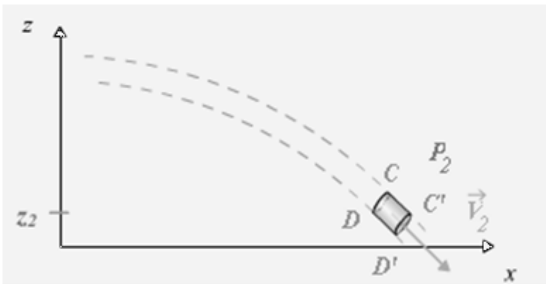
1. The first term represents the work of the pressure forces (per unit volume).
2. The second term represents the kinetic energy (per unit volume).
3. The third term represents the potential energy of the situation (per unit volume).

Second approach: Conservation of energy

2.3 Kinetic energy theorem

As we have seen previously, the Bernoulli relation is an equation of conservation of the mechanical energy of the fluid during its movement, let us see how to find the result using the kinetic energy theorem.





The assumptions regarding the fluid and the flow are the same. We consider a current stream ABA'B' at time t. At t + dt, the stream changes to CDC'D'.

Appling the kinetic energy theorem to the current flow between times t and t + dt:

$$\frac{1}{2} dm(V_2^2 - V_1^2) = dm\, g(z_1 - z_2) + (P_1 - P_2)d\tau$$

dτ = ABB'A' = CDC'D' and dm = ρdτ

We therefore obtain:

$$\frac{1}{2}\rho V_1^2 + \rho g z_1 + P_1 = \frac{1}{2}\rho V_2^2 + \rho g z_2 + P_2$$

We find the three terms indicating the conservation of the mechanical energy of the fluid: kinetic energy, potential situational energy and pressure energy (always per unit of volume).

Other writings of the Bernoulli equation

Bernoulli's equation can be written in other forms:

By dividing all the terms par ρ, the unit of the different terms of the equation becomes the joule per kilogram:

$$\boxed{\frac{1}{2}V_1^2 + g z_1 + \frac{P_1}{\rho} = cst\ (\mathrm{J.kg^{-1}})}$$

By dividing all the terms of the equation by ρg, the unit of the different terms becomes the meter:

$$\boxed{\frac{V_1^2}{2g} + z_1 + \frac{P_1}{\rho g} = H = cst\ (m)}$$

2.4 Graphic interpretation of Bernoulli equation

In the language of fluid mechanics, the mechanical energy of the fluid represented by the sum of the three terms of the Bernoulli relation is called the flow head.Writing the Bernoulli relation in meters shows that we can make a graphical interpretation:

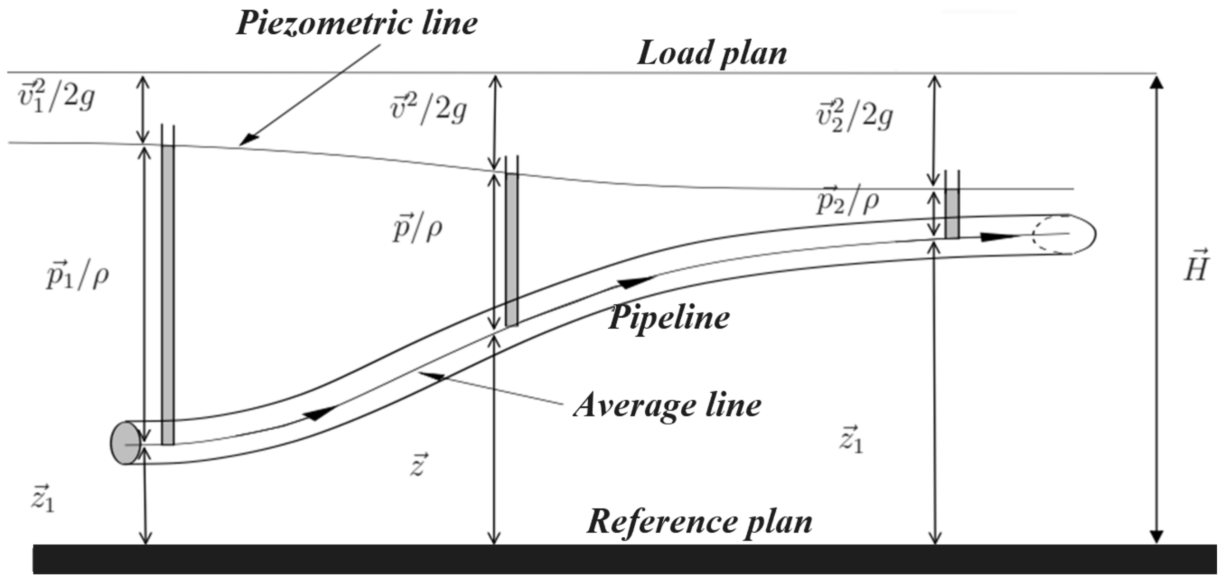


Fig . Graphical representation of the load of a flow in a conduit.

Z: The elevation head is (*h*) of the fluid above an arbitrarily designated zero point: (compared to a reference plan).

P / ρg pressure head is the height of a liquid column that corresponds to a particular pressure exerted by the liquid column on the base of its container. It may also be called **static pressure head** or simply **static head**.

V^2 / 2g :velocity head is due to the bulk motion of a fluid.

Note: Case of real fluids (viscous fluid)

For real fluids (having a viscosity), the load line will not be horizontal but decreasing, this decrease will indicate the load losses in the flow field (energy losses).

Note: Case of gases

For velocity not exceeding 0.3 times the speed of sound, we can assume that ρ = cste. In addition, the energy related to variations in coasts is often negligible (compared to the other terms). We therefore neglect the term ρgz in the Bernoulli equation.

2.5 Generalized Bernoulli theorem

Hypotheses

We consider the flow of an **incompressible fluid** in a non-steady state in a volume τ. We call F the volume force exerted by the moving walls of a machine on the fluid..

The fundamental principle of dynamics is written:

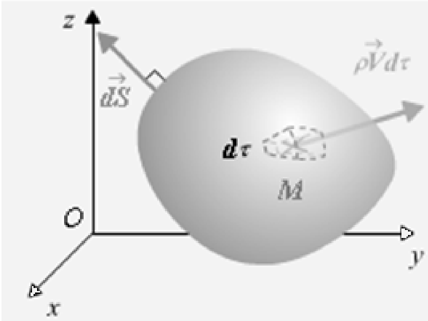


Fig 2.2 volume and surface forces applied by control volume

$$\sum \vec{F} - \overrightarrow{grad}P = \rho\,\vec{a}$$

Either by developing:

$$-\overrightarrow{grad}\left(P + \rho g z + \frac{\rho V^2}{2}\right) + \vec{F} = \rho\frac{\partial \vec{V}}{\partial t} + \rho\overrightarrow{rot}\vec{V}\wedge\vec{V}$$

$$\iiint_\tau \vec{V}.\overrightarrow{grad}\left(P + \rho g z + \frac{\rho V^2}{2}\right)d\tau + \iiint_\tau \frac{\rho}{2}\frac{\partial V^2}{\partial t}d\tau = \iiint_\tau \vec{F}.\vec{V}d\tau$$

Avec l'égalité vectorielle $div(f.\vec{A}) = f div(\vec{A}) + \vec{A}\,\overrightarrow{grad}\,f$ true for a scalar function f and a vector field $\vec{A}$) applied to:

$$f = \left(P + \rho g z + \frac{\rho V^2}{2}\right) \text{ and } \vec{A} = \vec{V}$$

We obtain : $div\left(\left(P + \rho g z + \frac{\rho V^2}{2}\right)\vec{V}\right) = \vec{V}\,\overrightarrow{grad}\left(P + \rho g z + \frac{\rho V^2}{2}\right)$

$$\iiint div\left(\left(P + \rho g z + \frac{\rho V^2}{2}\right)\vec{V}\right)d\tau + \iiint_\tau \frac{\rho}{2}\frac{\partial V^2}{\partial t}d\tau = \dot{W}$$

$$\boxed{\underbrace{\oiint_S \left(\left(P + \rho g z + \frac{\rho V^2}{2}\right)\vec{V}\right)dS}_{Energy\ flow\ out\ of\ S} + \iiint_\tau \frac{\rho}{2}\frac{\partial V^2}{\partial t}d\tau = \underbrace{\dot{W}}_{Power\ supplied\ to\ the\ fluid}}$$

2.6 Fluid passing through a hydraulic machine

Hypotheses

Let us consider a perfect fluid, in permanent flow, at the level of the machine. The walls provide the fluid with a volumetric mechanical energy W ($J.m^{-3}$),

heat exchanges are neglected.

We consider at time t a current trickle ABB'A' of an incompressible fluid, in the gravity field and passing through the hydraulic machine.

At time t+dt, the fluid is in CDD'C'.

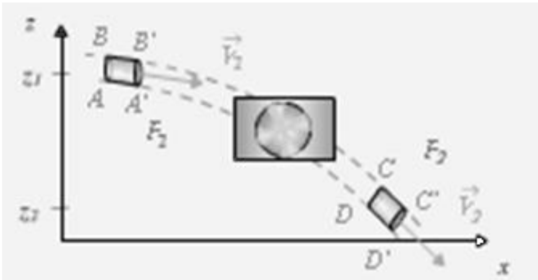


Fig 2.3 Flow through a hydraulic machine

Applying the kinetic energy theorem to the current flow between the instants t and t t+dt

$$\frac{1}{2}dm(V_2^2 - V_1^2) = dm\ g(z_1 - z_2) + (P_1 - P_2)d\tau + Wd\tau$$

With $\rho d\tau$ we obtain after simplification:

$$\frac{1}{2}\rho V_1^2 + \rho g z_1 + P_1 = \frac{1}{2}\rho V_2^2 + \rho g z_2 + P_2 + W \quad \begin{vmatrix} W < 0\ Machine\ motrice \\ W > 0\ Machine\ géneratice \end{matrix}$$

Interpretation:
The energy of the fluid after the machine is equal to that before it plus the energy provided by the machine

2.7 non-permanent flows and rotational flow

We consider the flow of a perfect fluid, the acting volume forces derive from a potential, and the flow is non-permanent.

We write the Euler equation:

$$\sum \vec{F} - \overrightarrow{grad}P = \rho\,\vec{a}$$

En tenant compte des hypothèses, et en développant l'expression de l'accélération, on obtient :

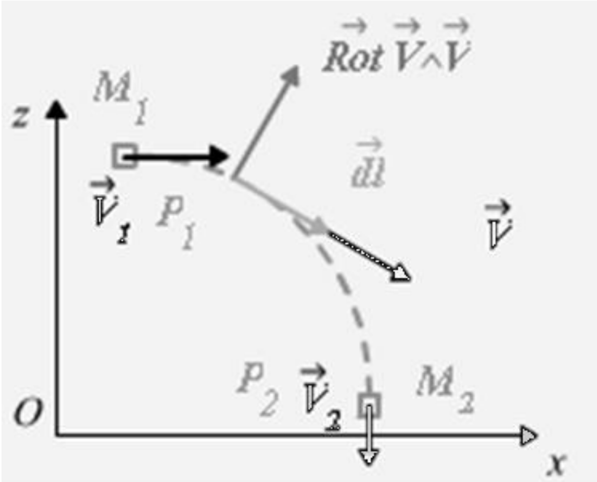


Fig 2.4 velocity field

$$\overrightarrow{grad}\left(P+\rho gz+\frac{\rho\, V^2}{2}\right)+\rho\frac{\overrightarrow{\partial V}}{\partial t}=\rho\,\overrightarrow{rot}\vec{V}\wedge\vec{V}$$

We calculate the circulation of the previous expression between M_1 and M_2.

Demonstration:

$$\frac{1}{2}\rho V_2^2+\rho g z_2+P_2-\frac{1}{2}\rho V_1^2-\rho g z_1-P_1+\rho\int_{M_1}^{M_2}\frac{\overrightarrow{\partial V}}{\partial t}dl=0$$

$$\frac{1}{2}\rho V_2^2+\rho g z_2+P_2+\rho\int_{M_1}^{M_2}\frac{\overrightarrow{\partial V}}{\partial t}dl=\frac{1}{2}\rho V_1^2+\rho g z_1+P_1$$

Particular case

Case where the cross-section of the current flow is constant:

The fluid is incompressible, so the flow rate is conserved in volume$(div\overrightarrow{V}=0)$, as the section is constant, the velocity is also constant: $\overrightarrow{V_1}=\overrightarrow{V_2}$. At every moment$\vec{V}$et $\frac{\overrightarrow{\partial V}}{\partial t}$have the same value along the streamline.

$$\rho g z_2+P_2+\rho l\frac{\partial\vec{V}}{\partial t}=\rho g z_1+P_1$$

2.8 Momentum theorem

The particle derivative of the torsor $[Q]$ of the momentum of a material system is equal to the torsor of the external forces applied to this system.

Let be the torsor of the external forces, the mathematical translation of the statement is:

$$\frac{d[Q]}{dt} = [F_e]$$

Demonstration

The torsor $[Q]$ of the quantities of motion is written: $[Q] = \iiint_\tau \rho \vec{V} d\tau$

$\overrightarrow{\rho V} d\tau$ is the elementary torsor of the quantities of motion.

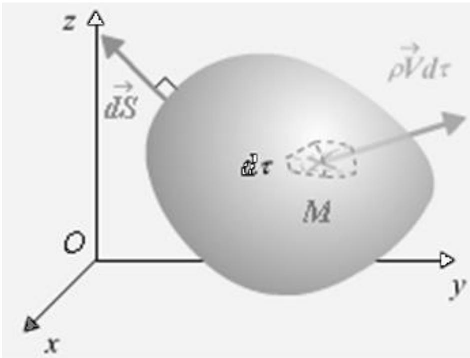


Fig 2.5 motion of control volume

Develop the particle derivative of the torsor of the quantities of motion using the result obtained in fluid kinematics.$\frac{d[Q]}{dt} = \iiint_\tau \frac{\partial(\rho\vec{V})}{\partial t} d\tau + \oiint_S (\rho\vec{V}) \vec{V}\, \vec{n}\, dS$

The torsor of external forces consists of two terms:

The torso of superficial surfaces:$[F_S] = \oiint_S [\vec{\sigma}] dS$

Torsor of volume forces: $[F_V] = \oiiint_\tau \rho\vec{F} d\tau$

Where

$$\iiint_\tau \frac{\partial(\rho\vec{V})}{\partial t} d\tau + \oiint_S (\rho\vec{V}) \vec{V}\, \vec{n}\, dS = \oiint_S [\vec{\sigma}] dS + \oiiint_\tau \rho\vec{F} d\tau$$

The equality of the resulting moments:

$$\boxed{\iiint_\tau \overrightarrow{OM} \wedge \frac{\partial(\rho\vec{V})}{\partial t} d\tau + \oiint_S \overrightarrow{OP} \wedge \rho\vec{V} \left(\vec{V}\, \vec{n}\right) dS = \oiint_S \overrightarrow{OP} \wedge \vec{\sigma}\, dS + \oiiint_\tau (\overrightarrow{OM} \wedge \rho\overrightarrow{F}) d\tau}$$

These equalities are general, true whatever the fluid and its motion.

2.9 Steady flow: Euler's theorem

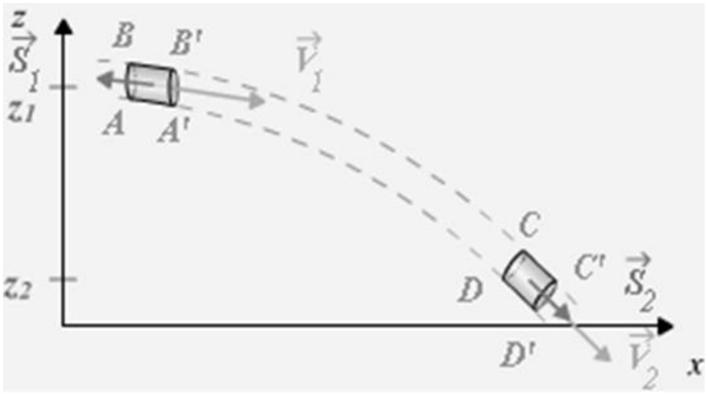
Hypotheses

A steady flow of fluid is considered (compressible or not).$\frac{\partial(\rho\vec{V})}{\partial t} = 0$

$$\oiint_S (\rho\vec{V})\vec{V}\,\vec{n}\,dS = \oiint_S \vec{\sigma}dS + \oiiint_\tau \rho\vec{F}d\tau$$

Resulting from the superficial surfaces:$[\vec{R}] = \oiint_S [\vec{\sigma}]dS$

Resultant of volume forces: $[\vec{P}] = \oiiint_\tau \rho\vec{F}d\tau$



$$q_m(\vec{V}_2 - \vec{V}_1) = \vec{P} + \vec{R}$$

$$\oiint_S (\rho\vec{V})\vec{V}\,\vec{n}\,dS = \rho_2 V_2 S_2 \overrightarrow{V_2} - \rho_1 V_1 S_1 \overrightarrow{V_1} = q_m(\vec{V}_2 - \vec{V}_1)$$

2.9.1 Theorem statement

In steady state, the system of flow rates of movements leaving the surface S is equivalent to the system of forces applied to the fluid contained in the surface.

Summary: To apply the momentum theorem:

- Define the surface limiting the fluid.

- Balance the pressure forces on the surface.

- Balance the volume forces**.**

Chapter 3: Real Fluid Dynamics

3. Local equation

Projecting the equation representing the equality of the resultants onto three trirectangular axes, we obtain three integral equations

$$\frac{d}{dt}\iiint_{\tau} \rho u_i \, d\tau = \oiint_{S} [\vec{T_i}] dS + \oiiint_{\tau} \rho F_i d\tau$$

Where u_i, F_i, T_i are respectively the projections of the velocity, the force volume density and the surface force.

The objective is to transform this formula to obtain a local equation (true at each point of the fluid) representing the dynamic equation.

$$\frac{d}{dt}\iiint_{\tau} \rho u_i \, d\tau = \iiint_{\tau} \frac{d}{dt}(\rho u_i) \, d\tau$$

$$\oiint_{S} [\vec{T_i}] dS = \oiint_{S} \sigma_{ij\, n_j} dS = \oiiint_{\tau} \frac{\partial \sigma_{ij}}{\partial x_j} d\tau$$

Note : $\frac{\partial \sigma_{ij}}{\partial x_j} = div \; \sigma_{ij}$

$$\iiint_{\tau} \frac{d}{dt}(\rho u_i) \, d\tau = \oiiint_{\tau} \rho F_i d\tau + \oiiint_{\tau} \frac{\partial \sigma_{ij}}{\partial x_j} d\tau$$

$$\frac{d}{dt}(\rho u_i) = \rho F_i + \frac{\partial \sigma_{ij}}{\partial x_j}$$

It is assumed that the volume forces are derived from a potential:
$\vec{F} = -\overrightarrow{grad}\, U$ donc $F_i = -\frac{\partial U}{\partial x_i}$

Let us introduce the following expression:

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}$$

$$\frac{d}{dt}(\rho u_i) = -\rho \frac{\partial U}{\partial x_i} - \frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

We obtain:

$$\boxed{\frac{d}{dt}(\rho\vec{V}) = -\,\rho\overrightarrow{grad}U - \overrightarrow{grad}P + \vec{f}}$$

This equation is the fundamental equation of viscous fluid dynamics.

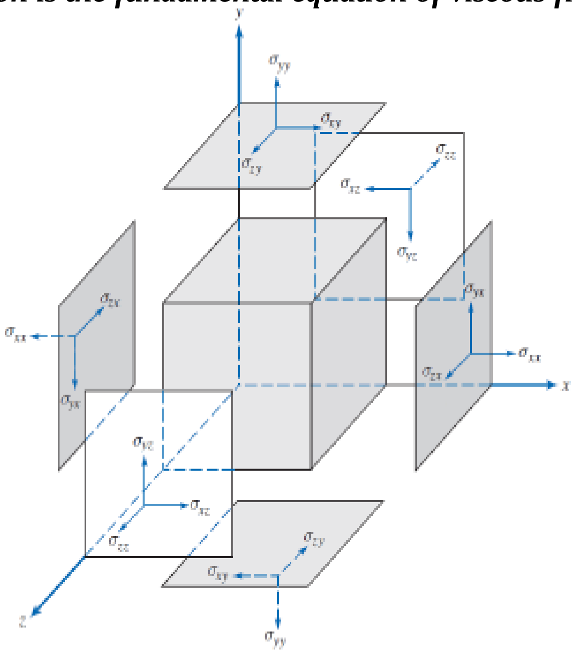


Fig 2.6 Stress tensor

Let us write a limited development of the velocity to the first order. The gradient operator will be noted ∇ (we also find the notation "grade"):

$$\nabla\vec{U} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

The velocity gradient $\boldsymbol{\nabla U}$ is a tensor of ***order 2*** that can be expressed as a sum of a symmetric tensor d and an anti-symmetric tensor r:

$$\nabla\vec{U} = d + r \;\; avec \quad \begin{cases} d = \frac{1}{2}.\left[\nabla\vec{U} + \left(\nabla\vec{U}\right)^T\right] \\ r = \frac{1}{2}.\left[\nabla\vec{U} - \left(\nabla\vec{U}\right)^T\right] \end{cases} \quad \text{Soit :} \quad \begin{matrix} d_{ij} = \frac{1}{2}.\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \\ r_{ij} = \frac{1}{2}.\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right] \end{matrix}$$

The tensor d is called the strain rate tensor and the tensor r the rotation rate tensor.

3.1 Deformations

In fluid mechanics, deformation rates are usually characterized by the tensor $\gamma^{\cdot}$ rather than the deformation rate tensor **d** defined by $\dot{\gamma}$:

$$\dot{\gamma} = 2.d \quad \text{So :} \quad \dot{\gamma}_{ij} = \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

In any case, it is natural to describe the total strain **(d)** as the sum of an isotropic expansion **(d_{exp})** and a constant volume shear **(d_{cis}):**

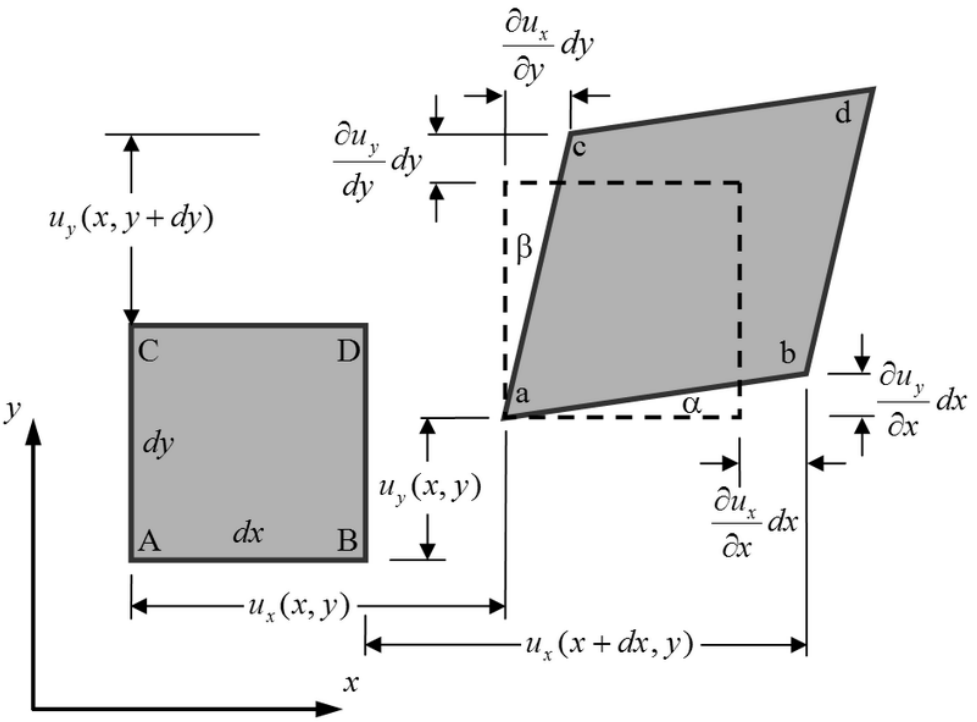
$$d = d_{exp} + d_{cis}$$

In the following, we analyze each of the two contributions

3.2 Expansion:

The following figure shows that during the time period dt, the volume of matter dV= dx,dy,dz evolves by the following amount:

Illustration of deformations +translation+expansion+shear+rotation



Two-dimensional geometric expansion of an infinitesimal material element

$$dV = \left(\frac{\partial u_x}{\partial x}.dx.dt\right).dy.dz + \left(\frac{\partial u_y}{\partial y}.dy.dt\right).dx.dz + \left(\frac{\partial u_z}{\partial z}.dz.dt\right).dx.dy$$

The relative volume variation is written as:

$$\frac{dV}{V} = \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) dt$$

$$\frac{dV}{V} = (div\ \vec{U})dt$$

Or again:

$$div\ \vec{U} = \frac{1}{V} . dV\ dt$$

An incompressible fluid is characterized by $\boldsymbol{div\ \vec{U} = 0}$

The component of the strain rate tensor d associated with the isotropic expansion will be denoted $\boldsymbol{d_{exp}}$ and, logically, proportional to the divergence of the velocity. Since we are working in three dimensions, we will have:

$$\boldsymbol{d_{exp} = \frac{trace(d)}{3} . I} \qquad \textbf{Soit :}\ \boldsymbol{d_{exp,ij} = \frac{div\,\vec{v}}{3} . \delta_{ij}}$$

Cisaillement On note d_{cis} le tenseur obtenu par différence entre d et d_{exp}

$$d_{\mathrm{cis}} = d - d_{\mathrm{exp}}$$

Either:

$$d_{\mathrm{cis}} = \frac{1}{2} . \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] - \frac{div\ \vec{v}}{3} . \delta_{ij}$$

3.2 Newtonian fluid

We have seen that the strain rate tensor $\boldsymbol{d}$ can be written as the sum of two contributions: a constant volume shear d_{cis} and an isotropic expansion d_{exp}:

$$d = d_{\mathrm{cis}} + d_{\mathrm{exp}}$$

The shear stress tensor can also be seen as the sum of these two contributions:

$$\tau = \tau_{\mathrm{cis}} + \tau_{\mathrm{exp}}$$

A Newtonian fluid is a fluid characterized by a simple proportionality relationship between $\boldsymbol{\tau_{\mathrm{cis}}}$ and $\boldsymbol{d_{\mathrm{cis}}}$ on the one hand and between $\boldsymbol{\tau_{\mathrm{exp}}}$ and $\boldsymbol{d_{\mathrm{exp}}}$ on the other hand:

$$\begin{cases} \tau_{\mathrm{cis}} = 2\mu . d_{cis} \\ \\ \tau_{\mathrm{exp}} = 3\mu_v . d_{exp} \end{cases}$$

Finally, the shear stress tensor of a Newtonian fluid is written as:

$$\tau_{ij} = \mu . \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] + \underbrace{\left(\mu_V - \frac{2}{3}\mu\right)}_{\eta} . div\,\overrightarrow{U} . \delta_{ij}$$

$$\tau_{ij} = 2\mu\ D_{ij} + \eta\left(\frac{\partial u_i}{\partial x_j}\right)$$

$$\frac{d}{dt}(\rho u_i) = -\rho\frac{\partial U}{\partial x_i} - \frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\frac{\partial \tau_{ij}}{\partial x_j} = \mu \Delta u_i + (\mu + \eta)\frac{\partial g}{\partial x_i}$$

We recall:

$$e = div\vec{V}$$

$$\boxed{\begin{cases} \dfrac{d}{dt}(\rho u) = -\rho\dfrac{\partial U}{\partial x} - \dfrac{\partial P}{\partial x} + \mu \Delta u + (\mu + \eta)\dfrac{\partial g}{\partial x} \\ \dfrac{d}{dt}(\rho v) = -\rho\dfrac{\partial U}{\partial y} - \dfrac{\partial P}{\partial y} + \mu \Delta v + (\mu + \eta)\dfrac{\partial g}{\partial y} \\ \dfrac{d}{dt}(\rho w) = -\rho\dfrac{\partial U}{\partial z} - \dfrac{\partial P}{\partial z} + \mu \Delta w + (\mu + \eta)\dfrac{\partial g}{\partial z} \end{cases}}$$

With

$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$g = div\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

In vector form, the Navier Stokes equation is:

$$\boxed{\frac{d}{dt}\left(\rho\vec{V}\right) = -\rho\overrightarrow{grad}U - \overrightarrow{grad}P + \mu\Delta u + (\mu+\eta)\overrightarrow{grad}(div\,\vec{V})}$$

Last hypothesis:

On suppose le fluide imcompressible donc $div\,\vec{V} = 0$

where:

$$\rho\frac{d\vec{V}}{dt} = -\rho\overrightarrow{grad}U - \overrightarrow{grad}P + \mu\Delta u$$

3.3 **Flow in a two-dimensional conduit**

Hypotheses

We consider an infinite two-dimensional conduit, of small thickness 2b, the x axis coincides with the axis of symmetry of the conduit.

The fluid is considered Newtonian, incompressible in stationary (permanent) flow along the x axis.

The Navier Stockes equation gives:

$$\frac{d}{dt}(\rho u) = -\rho\frac{\partial}{\partial x}(\rho g y) - \frac{\partial P}{\partial x} + \mu\Delta u$$
$$\frac{d}{dt}(\rho v) = -\rho\frac{\partial}{\partial y}(\rho g y) - \frac{\partial P}{\partial y} + \mu\Delta v$$

u, v components of the velocity along x and y

Symmetry: in any plan xoy, we have the same profile so w=0

There are no components of the velocity along y so v=0

The continuity Equation $\mathbf{div}\,\vec{\mathbf{V}} = \mathbf{0}$ imposed $\frac{\partial u}{\partial x} = \mathbf{0}$

Differential equation

$$\begin{cases} 0 = -\dfrac{\partial P}{\partial x} + \mu \Delta u \\ 0 = -\dfrac{\partial P}{\partial y} - \rho g \end{cases}$$

Since the thickness is small, we can consider that the term representing the gravitational forces is negligible.

The velocity profile in the duct is therefore given by solving the following differential equation:

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial P}{\partial x}$$

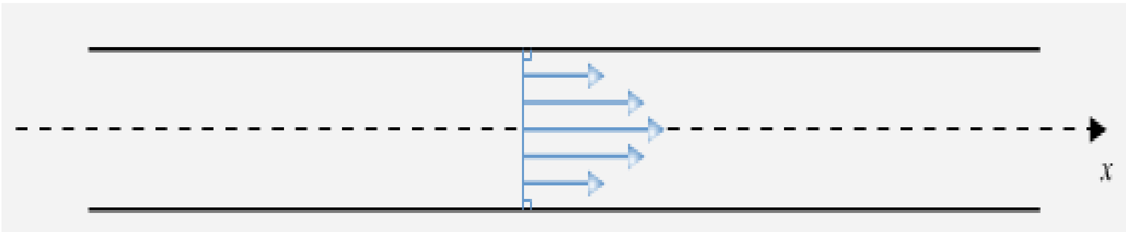
Since u only depends on y, we can replace the partial derivative of the velocity by a total derivative.

$$\mu \frac{d^2 u}{dy^2} = \frac{\partial P}{\partial x}$$

With $\boldsymbol{\mu}$ et $\frac{\partial P}{\partial x}$ are the constants

Integrating twice we get:

$\boldsymbol{u(y) = \frac{1}{2\mu}\left(\frac{\partial P}{\partial x}\right)(y^2 - b^2)}$ **:The velocity profile is parabolic.**



3.4 Low Reynolds number- Stocks flow

Stokes formula If Re ≪ 100, the factor $C_x$varies significantly according to the **law:**Formule de Stokes Si Re ≪ 100, le facteur C varie sensiblement suivant la loi :

$$lnC_x = -lnRe + Cst \quad \rightarrow \ C_x \, Re = \ Cst$$

Therefore, the drag is written as:

$$\boxed{T = C_x \frac{\rho_{f} \, v^2}{2} \, S = \left(\frac{Cst}{Re}\right) \frac{\rho_{f} \, v^2}{2} \, S = Cst \times \frac{S}{2D} \mu v}$$

For a sphere of radius r, we obviously have $S = \pi \, r^2$et $= 2r$, we further show that the numerical constant is equal in this case to 24. This results in the following expression for the drag.

It is instructive to express this force T as a function of the low Reynolds number

$$Re = \frac{r \, v \, \rho}{\mu} :$$

$$\boxed{T = 6\pi\mu \, r \, v = 6\pi Re \, \frac{\mu^2}{\rho}}$$

<u>**Note:**</u> For even lower Reynolds numbers, we can neglect the acceleration term of a body compared to the viscosity force, which gives the following relation, by designating by F an additional force, for example the weight of this body:

$$\boxed{m \frac{dv}{dt} = -\alpha \, v + F_s \approx 0 \quad v \approx \frac{F_s}{\alpha}}$$

Chapter 4: Boundary Layers

4. 1Introduction

The concept of boundary layer was first introduced by a German engineer, Prandtl, in 1904. According to Prandtl's theory

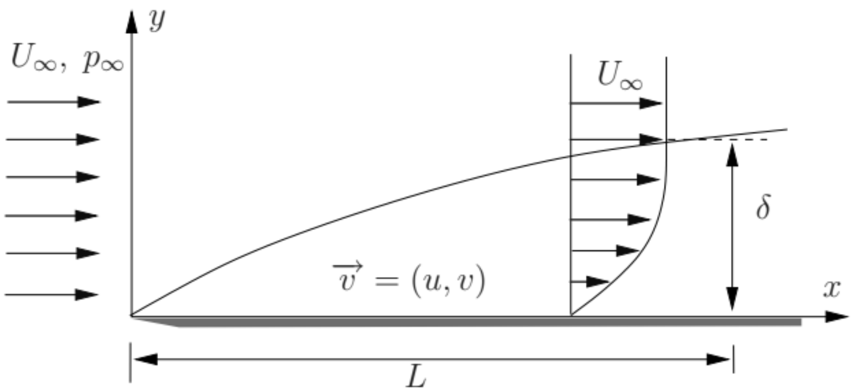


Fig 4.1 Boundary layer on a flat plate

When a real fluid flows past a fixed solid wall, the flow is divided into two regions.A thin layer in the vicinity of the solid wall where viscous forces and rotation cannot be neglected.

An external region where viscous forces are very small and can be neglected. The flow behavior is similar to the free flow upstream.

The flow on the plate can be divided into two domains.

a. **$0 \leq y \leq \delta$** the viscous force effect is important.
b. **$y > \delta$:** Flow region external to the boundary layer where the viscous force is very small and can be neglected. There is no velocity gradient in this region and the fluid particle does not rotate when it enters the region external to the boundary layer. Hence, the flow is also called irrotational flow.

Let's return to the boundary layer problem

The pressure p and time are assumed to be of order ρU_∞^2 et $L/_{U_\infty}$ respectively.

We denote $y = 0$ the flat plate and et $e = LU_\infty/\nu$.

It is advantageous to start by considering the mass conservation equation. Here we assume that $\frac{\partial u}{\partial x}$ is of the order$\frac{U_\infty}{L}$ and $\frac{\partial v}{\partial y}$ is of the order $\frac{V_0}{\delta}$, being a characteristic scale for v. To satisfy the mass conservation equation, these two terms must be of the same order of magnitude. Then, we deduce:

$$\underbrace{\frac{\partial u}{\partial x}}_{\frac{U_\infty}{L}} - \underbrace{\frac{\partial v}{\partial y}}_{\frac{V_0}{\delta}} = 0$$

$$\boxed{V_0 = \frac{\delta U_\infty}{L}}$$

With this estimate taken into account, let us now examine the orders of magnitude of the different terms of the momentum equation:

$$\underbrace{\frac{\partial u}{\partial t}}_{\substack{\frac{U_\infty{}^2}{L}\\ \propto\\ 1}} + \underbrace{u\frac{\partial u}{\partial x}}_{\substack{\frac{U_\infty{}^2}{L}\\ \propto\\ 1}} + \underbrace{v\frac{\partial u}{\partial y}}_{\substack{\frac{\delta U_\infty U_\infty}{L\quad \delta}\\ \propto\\ 1}} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial x}}_{\substack{\frac{1\rho U_\infty{}^2}{\rho\quad L}\\ \propto\\ 1}} + \underbrace{\nu\frac{\partial^2 u}{\partial x^2}}_{\substack{\frac{\nu U_\infty}{L^2}\\ \propto\\ \frac{1}{Re}}} + \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\substack{\frac{\nu U_\infty}{\delta^2}\\ \propto\\ \frac{1}{Re\left(\frac{L}{\delta}\right)^2}}}$$

According to y:

$$\underbrace{\frac{\partial v}{\partial t}}_{\substack{\frac{\delta U_\infty{}^2}{L^2}\\ \propto\\ 1}} + \underbrace{u\frac{\partial v}{\partial x}}_{\substack{\frac{\delta U_\infty{}^2}{L^2}\\ \propto\\ 1}} + \underbrace{v\frac{\partial v}{\partial y}}_{\substack{\frac{\delta U_\infty \delta U_\infty}{L\quad L\delta}\\ \propto\\ 1}} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial x}}_{\substack{\frac{1\rho U_\infty{}^2}{\rho\quad \delta}\\ \propto\\ 1}} + \underbrace{\nu\frac{\partial^2 v}{\partial x^2}}_{\substack{\frac{\nu\delta U_\infty}{L^3}\\ \propto\\ \frac{1}{Re}\left(\frac{\delta}{L}\right)}} + \underbrace{\nu\frac{\partial^2 v}{\partial y^2}}_{\substack{\frac{\nu\delta U_\infty}{L\delta^2}\\ \propto\\ \frac{1}{Re}\left(\frac{L}{\delta}\right)}}$$

Before applying the principle of least degeneracy we are faced with two possibilities:

a. δ is (L) which leads for Re large to neglect the two terms where viscosity intervenes.
b. δ is small compared to L which allows to keep the term in $\nu\frac{\partial^2 u}{\partial y^2}$ in the momentum equation where viscosity intervenes. For this we must have:

$$\frac{1}{Re}\left(\frac{L}{\delta}\right) = 1 \quad \Rightarrow \qquad \boxed{\delta \sim Re^{-1/2}\, L}$$

With this choice for δ the equations reduce to:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial P}{\partial y} = 0$$

At the leading order in Re. This system of equations is called the Prandtl equation for the boundary layer.

For the two-dimensional problem on a flat plate these equations are written:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + v\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial P}{\partial y} = 0$$

As for the pressure, it is determined from the flow outside the boundary layer $\vec{v} = (U_e(x,t), 0)$:

$$\boxed{\frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}}$$ Boundary layer equation

4.2 Boundary layer development

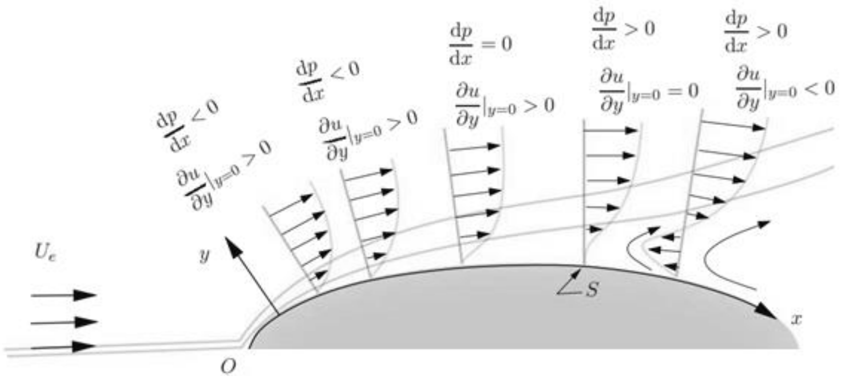
The boundary layer equation shows that the curvature of the velocity profile at the wall, with u(x, y = 0) = v(x, y = 0) = 0, depends only on the pressure gradient:

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + v\left(\frac{\partial^2 u}{\partial x^2}\right)_{y=0}$$

This leads to:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{y=0} = -\frac{1}{v}U_e\frac{dU_e}{dx}$$

Because U_e and P only depend on x.



4.2 Evolution of the boundary layer on a solid body, x is the curvilinear abscissa, S the point of detachment or separation.

A negative pressure gradient, dp/dx < 0, implies an increase in velocity Ue, dUe/dx > 0, an acceleration in the direction of flow.

$\frac{dU_e}{dx} > 0$ i.e. an acceleration in the direction of the flow. On the other hand, a positive gradient leads to a deceleration of the flow. This is why the pressure gradient in the first case is said to be favorable and in the second unfavorable.

To solve this problem, let us start by integrating the mass conservation equation with respect to y. We find:

$$\frac{\partial}{\partial x}\int_0^y udy + v(x, y = 0) = 0$$

Which shows that there exists a function ψ(x, y) defined by:

$$\Psi(x, y) = \int_0^y udy$$

This satisfies the mass conservation equation:

$$u = \frac{\partial \Psi}{\partial y} \qquad v = -\frac{\partial \Psi}{\partial x} \qquad tel\ que\ \ \Psi(x, y = 0) = 0$$

So let's look for a solution in the form:

$$\Psi(x, y) = \sqrt{vx\, U_\infty}\, f(n) \qquad with\ \ y = \ \eta\sqrt{vx\,/U_\infty}$$

For laminar flow past the plate, the boundary layer equations can be solved exactly for u and ψ, assuming that the free-stream velocity U is constant (dU/dx = 0). The solution was given by Prandtl's student Blasius, in his 1908 dissertation from Göttingen. With an ingenious coordinate transformation, Blasius showed that the dimensionless velocity profile u/U is a function only of the single composite dimensionless variable

$$\frac{\partial}{\partial x} = \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta},\ \ \frac{\partial}{\partial y} = \left(\frac{U_\infty}{v\, x}\right)^{1/2}\frac{\partial}{\partial \eta} \quad with\ \ \frac{\partial \eta}{\partial x} = -\frac{\eta}{2x}$$

This leads, with the previous relations, to:

$$\left.\begin{aligned} u &= \frac{\partial \Psi}{\partial y} = U_\infty\, f'(\eta) \\ v &= -\frac{\partial \Psi}{\partial x} = \frac{1}{2}\left(\frac{vU_\infty}{x}\right)^{\frac{1}{2}}(\eta f'(\eta) - f(\eta)) \end{aligned}\right\}$$

Where the prime denotes differentiation with respect to η. Substitution obove equation into the boundary layer equations , reduces the problem, after much algebra, to a single third-order nonlinear ordinary differential equation for f:

$$\boxed{f''' + {}^1\!/_2\, f\, f'' = 0 \qquad Blasius\ solution}$$

The boundary conditions become:

$\boxed{at\ y = 0,\ \ f = 0,\ f' = 0}$ $\boxed{\eta = \infty,\ f' = 1}$

This is the Blasius equation, for which accurate solutions have been obtained only by numerical integration.

4.3 Laminar flow

Since u/U approaches to 1.0 only as $\rightarrow y_{\infty}$, it is customary to select the boundary layer thickness δ as that point where $u/U = 0.99$. δ occurs at $\eta \approx 5$

$$\boxed{\delta_{99\%}\left(\frac{U}{v_x}\right)^{1/2} \approx 5 \qquad \frac{\delta}{x} \approx \frac{5}{Re_x^{1/2}}}$$

With the profile known, Blasius, of course, could also compute the wall shear and displacement thickness:

$$\boxed{C_f = \frac{0.664}{Re_x^{1/2}} \qquad \frac{\delta^*}{x} = \frac{1.721}{Re_x^{1/2}}}$$

Notice how close these are to our integral estimates. When C_f is converted to dimensional form, we have:

$$\boxed{\tau_w(x) = \frac{0.332\ \rho^{1/2}\mu^{1/2}U^{1.5}}{x^{1/2}}}$$

The drag increases only as the square root of the plate length. The nondimensional drag coefficient is defined as:

$$\boxed{C_D = \frac{2D(L)}{\rho\ U^2\ b\ L} = \frac{1.328}{Re_x^{1/2}} = 2C_f(L)}$$

Thus, for laminar plate flow, CD equals twice the value of the skin friction coefficient at the trailing edge. This is the drag on one side of the plate. Kármán pointed out that the drag could also be computed from the momentum relation . In dimensionless form becomes:

$$\boxed{C_D = \frac{2}{L}\int_{\mathbf{0}}^{\delta} \frac{\boldsymbol{u}}{\boldsymbol{U}}\left(\mathbf{1} - \frac{\boldsymbol{u}}{\boldsymbol{U}}\right)\ \boldsymbol{dy}}$$

This can be rewritten in terms of the momentum thickness at the trailing edge:

$$\boxed{C_D = \frac{2\delta_2(L)}{L}}$$

Computation of θ from the profile u/U or from CD gives

$$\boxed{\frac{\delta_2}{\boldsymbol{x}} = \frac{0.664}{Re_x^{1/2}} \qquad\qquad Laminar\ flat\ plat}$$

4.5 Turbulent Flow

There is no exact theory for turbulent flat-plate flow, although there are many elegant computer solutions of the boundary layer equations using various empirical models for the turbulent eddy viscosity.

Assuming δ = 0 at x = 0:

$$\boxed{Re_\delta \approx 0.16\, Re_x^{6/7}}$$

Thus the thickness of a turbulent boundary layer increases as $x^{6/7}$, far more rapidly than the laminar increase $x^{1/2}$.

The friction variation:

$$\boxed{C_f \approx \frac{0.027}{Re_x^{1/7}}}$$

The drag coefficient:

$$\boxed{C_D = \frac{0.031}{Re_L^{\frac{1}{7}}} = \frac{7}{6}\, C_f(L)}$$

Then CD is only 16 percent greater than the trailing-edge skin friction coefficient (compare with laminar flow).

4.6 Reynolds Number and Geometry Effects

In Figure(4.3) a uniform stream U moves parallel to a sharp flat plate of length L. If the Reynolds number UL/ν is low (Fig. 7.1a), the viscous region is very broad and extends far ahead and to the sides of the plate. The plate retards the oncoming stream greatly, and small changes in flow parameters cause large changes in the pressure distribution along the plate

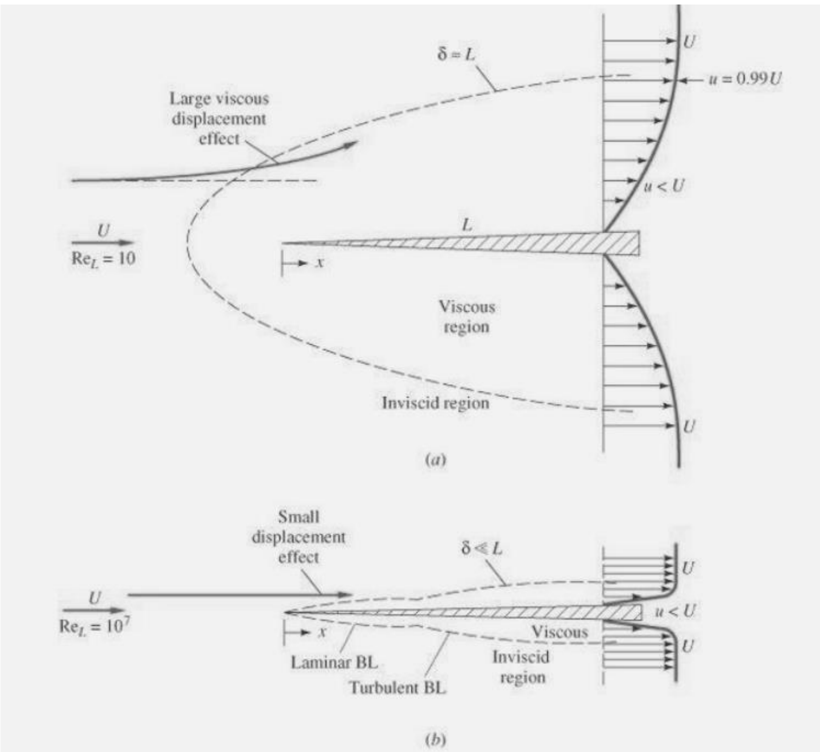


Fig. 4.3 Comparison of flow past a sharp flat plate at low and high Reynolds numbers: (a) laminar, low-Re flow; (b) high-Re flow.

4.3 The thickness of the boundary layer

For flow along a flat plate this definition translates to:

$$\boxed{\delta = \delta_{0.99} \simeq 5\, x\, Re^{-1/2}} \qquad \text{Blasius equation (1908)}$$

We define the boundary layer thickness δ as the locus of points where the velocity u parallel to the plate reaches 99%of the external velocity U.

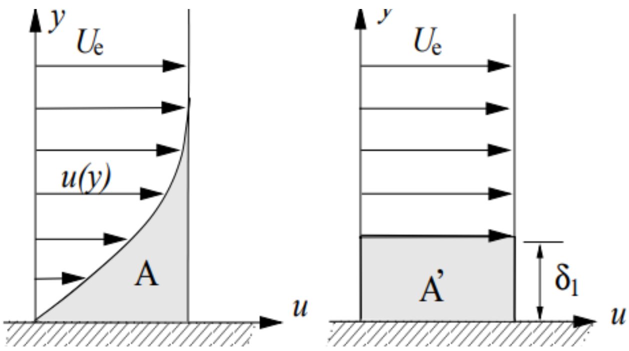


Fig 4.4 Definition of displacement thicknessδ_1

Areas A and A' are equal:

$$A = A' \int_0^{\delta} (U_e \ - u) \ dy$$

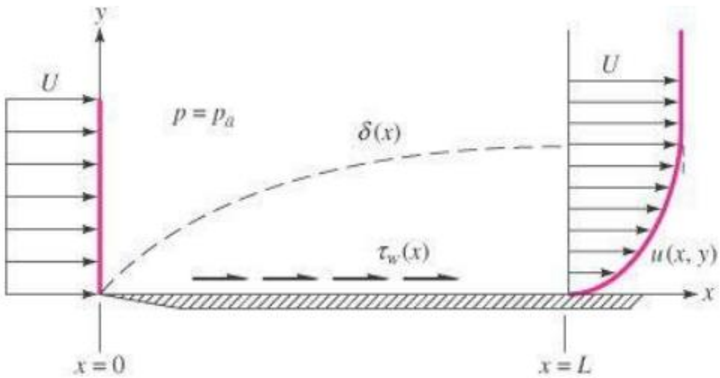
The displacement thickness then allows us to describe the flow deficit Qp − Qv as if the flow near the wall were in perfect fluid:

$$\int_0^{h\to\infty} (U_e \ - u) \ dy = \ U_e \ \ \delta_1$$

Et,

$$\delta_1 = \int_0^{h\to\infty} (1 - \frac{u}{U_e}) \ dy$$

Then, The quantity δ_1 is called the displacement thickness of the boundary layer. To relate it to $u(y)$



Kármán's Analysis of the Flat Plate

Equation below was derived in 1921 by Kármán, who wrote it in the convenient form of the momentum thickness δ2:

$$U^2 \ \delta_2 = \int_0^{h\to\infty} u(U \quad - u) \ \ dy$$

$$\delta_2 = \int_0^{\delta} \frac{u}{U} (1 - \frac{u}{U}) \ dy$$

Momentum thickness is thus a measure of total plate drag. Kármán then noted that the drag also equals the integrated wall shear stress along the plate:

and the thickness in energy δ3 :

$$U^2 \left(U \quad \delta_2\right) = \int_0^{h\to\infty} u(U^2 \ - u^2) \ \ dy$$

$$\delta_3 = \int_0^{h\to\infty} \frac{u}{U} (1 - \frac{u^2}{U^2}) \ dy$$

Useful formulations: These definitions allow us to draw the following formulas:

$$\boxed{\delta_2 = \alpha_1\delta,\ \alpha_1 = \int_0^1 f(1-f)d\eta, \qquad \delta_1 = \alpha_2\delta, \qquad \alpha_2 = \int_0^1 f(1-f)d\eta}$$

4.4 Von Karman integral equation

The Calculation results for the boundary layer on a flat plate at zero pressure gradient based on the theory of approximate solutions.
Reference: Schlichting, Boundary–Layer Theory, McGraw–Hill Book, New York (1966).

	$\frac{u}{U_\infty} = f(\eta)$	α_1	α_2	β_1	$\delta_1 \times \sqrt{\frac{U_\infty}{\nu x}}$	$\frac{\tau_p}{\mu U_\infty} \times \sqrt{\frac{\nu x}{U_\infty}}$	$C_x \times \sqrt{\frac{U_\infty \ell}{\nu}}$	$H = \frac{\delta_1}{\delta_2}$
1	$f(\eta) = \eta$	$\frac{1}{6}$	$\frac{1}{2}$	1	1.732	0.289	1.155	3.00
2	$f(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3$	$\frac{39}{280}$	$\frac{3}{8}$	$\frac{3}{2}$	1.740	0.323	1.292	2.70
3	$f(\eta) = 2\eta - 2\eta^3 + \eta^4$	$\frac{37}{315}$	$\frac{3}{10}$	2	1.752	0.343	1.372	2.55
4	$f(\eta) = \sin(\frac{1}{2}\pi\eta)$	$\frac{4-\pi}{2\pi}$	$\frac{\pi-2}{\pi}$	$\frac{\pi}{2}$	1.741	0.327	1.310	2.66
5	exacte	—	—	—	1.721	0.332	1.328	2.59

$$\beta_1 = f'(0), \qquad C_x\left(\frac{C_X l}{v}\right)^{1/2} = 2\delta_2\left(\frac{U_\infty l}{vx}\right)^{1/2}$$

$$\frac{d\delta_2}{dx} = \frac{\tau_p}{\rho U_e^2} - \frac{1}{U_e}\frac{dU_e}{dx}(2\delta_2 + \delta_1) = \frac{1}{2}C_f - \frac{\delta_2}{U_e}\frac{dU_e}{dx}(H+2)$$

Where $H = {}^{\delta_1}/_{\delta_2}$ is known as the shape parameter. This relation provides a differential equation for the boundary layer thickness provided that a suitable shape for the velocity profile is assumed.
Kármán arrived at what is now called the momentum integral relation for flat-plate boundary layer flow:

Chapitre 5 : Turbulent flow

5.1 Introduction in turbulent flow

Turbulent flows are part of everyday experience: the jet of water from the tap, the sills of a boat (if the speed is sufficient), the flows around a car, etc.

Indeed, given the adhesion of the fluid to the wall, the turbulence disappears there. But by moving away from the wall, the macroscopic agitation movements can develop more and more freely, so that turbulent diffusion prevails over molecular (viscous) diffusion and subsequently turbulence intensifies the mixing (the spatial "homogenization") of the properties. Hence a more uniform speed distribution in a turbulent regime compared to the laminar regime.

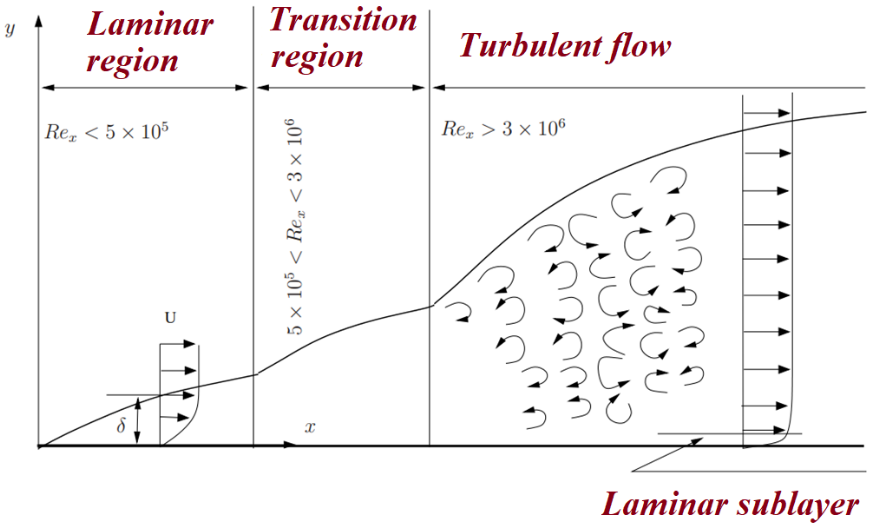


Fig 5.1 Development of the boundary layer on a flat plate.

5.2 Characteristics of turbulent flows

Turbulent flows arise when the driving force (or source of kinetic energy) that sets the fluid in motion is relatively intense compared to the viscosity forces that the fluid opposes to move. The driving force can take several forms:

• pressure gradients
• initial impulse for jets
• an Archimedean force (buoyancy) due to a temperature difference in the gravity field.

A turbulent flow leads to:

a. the reduction of kinematic, thermal, mass inhomogeneities within the flow, while increasing parital transfers. This is reflected in what is called turbulent diffusion;
b. the increase in viscous friction drag, possible reduction in form drag (related to pressure), by delaying possible separations;
c. promoting the mixing of a dispersed phase, but which can also cause the coalescence of droplets in two-phase flows.

Turbulent flows are characterized by different length scales:

a. Overall motion scale, L, corresponding to the "average" or "global" evolution of the flow,
b. Turbulent agitation motion scale, ℓ, reflecting vortices actually present in the flow,
c. Molecular agitation motion scale, l_m, reflecting only macroscopic effects in a continuous medium type approach

We define "turbulent diffusivities" which are a priori functions of the flow. This leads to defining:

a. a velocity scale $u^{'}$,
b. a length scale l from which we deduce the order of magnitude of a diffusivity by turbulent diffusion v_T :

$$v_{T \sim u' \times l}$$

Thus, the ratio to the diffusivity ν of the fluid: $\frac{v_T}{v} \sim \frac{u\prime \times l}{v} = Re_T$

Which is a Reynolds number of turbulence included between $10^2 < Re_T < 10^7$

To highlight the relative effect of turbulent diffusion versus molecular diffusion, we consider the boundary layer on a semi-infinite flat plate. Although the thickness of the laminar boundary layer δ(x) of Blasius is given by the parabolic evolution:

$$\delta(x) \approx 5\, x\, Re_x^{-1/2} = 5(v/U_e)^{1/2}\, x^{1/2}, \qquad Re_x = U_e\, x/v$$

The evolution of the thickness of the boundary layer in turbulent regime follows a thickening law in $x^{4/5}$

$$\delta(x) = 0.37\, x\, Re_x^{-1/5} = 0.37(v/U_e)^{1/5}\, x^{4/5}$$

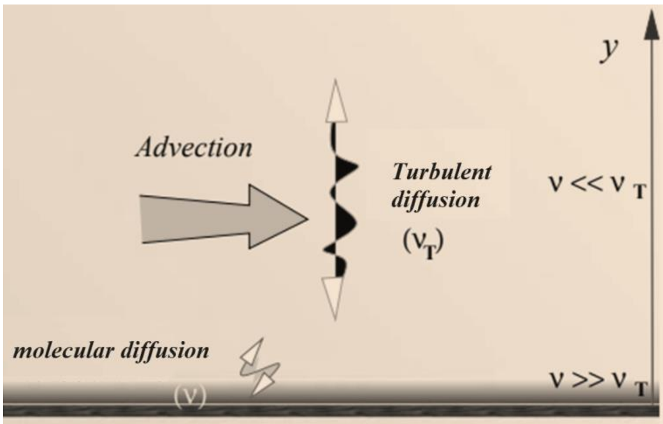


Fig 5.2 "turbulent diffusion acts as an "activator" of molecular diffusion,

5.4 Mean motion and fluctuations in incompressible flow

To describe turbulent motion it is convenient to decompose the motion into a mean motion into a fluctuation motion, or eddy motion.

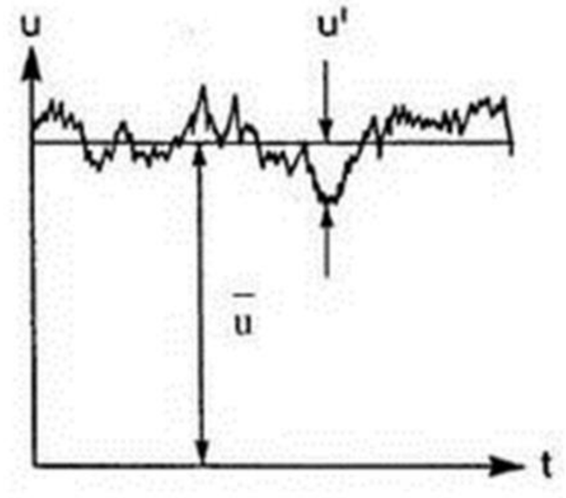


Fig 5.3 velocity signal mesured at given position

$$u = \bar{u} + u', \quad v = v + v', \qquad w = \bar{w} + w, \qquad P = \bar{P} + P'$$

In the case of compressible turbulent flow, it is also necessary to pose

$$\rho = \bar{\rho} + \rho', \qquad T = \bar{T} + T'$$

The time average is calculated at a fixed point in space and given, for example, by:

$$\bar{u} = \frac{1}{t_1}\int_{t_0}^{t_0+t_1} u\, dt$$

or the interval t_1 is long enough so that: $\overline{u'} = \overline{v'} = \overline{w'} = \overline{P'} = \overline{\rho'} = \overline{T'} = 0$

Before establishing the equations for the turbulent boundary layer it is useful to recall the rules to follow for the calculation of the average quantities

$$\bar{\bar{f}} = \bar{f}, \;\; \overline{f+g} = \bar{f} + \bar{g}, \quad \overline{\bar{f}.g} = \bar{f}.\bar{g}, \qquad \overline{\frac{\partial f}{\partial S}} = \frac{\partial\, \bar{f}}{\partial S}, \quad \overline{\int f\, dS} = \int \bar{f}\, dS$$

or S represents one of the variables x, y, z or t. Let us apply these formulas to flux quantities, for example, $\overline{v_i.v_j}$ ou $v_i = u, v \; or \; w$

$$v_i.v_j = (\bar{v}_i + v_i')(\bar{v}_j + v_j') = \bar{v}_i\,\bar{v}_j + \bar{v}_i v_j' + v_i'\bar{v}_j + v_i'v_j'$$

Which leads to:

$$\overline{v_i.v_j} = \bar{v}_i\,\bar{v}_j + \overline{v_i'v_j'}$$

5.5 Equations of motion and the Reynolds stress tensor

To determine the equations governing the mean motion we first take up the incompressible Navier–Stokes equations in Cartesian coordinate systems:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$
$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\Delta u$$
$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial y}+\mu\Delta v$$
$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\Delta w$$

Next, we multiply the continuity equation by ρu and then add the result to the equations above so we get:

$$\rho\left(\frac{\partial u}{\partial t}+\frac{\partial(u^2)}{\partial x}+\frac{\partial(v\,u)}{\partial y}+\frac{\partial(w\,u)}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\Delta u$$

In the same way he comes

$$\rho\left(\frac{\partial v}{\partial t}+\frac{\partial(u\,v)}{\partial x}+\frac{\partial(v^2)}{\partial y}+\frac{\partial wv}{\partial z}\right)=-\frac{\partial P}{\partial y}+\mu\Delta v$$
$$\rho\left(\frac{\partial w}{\partial t}+\frac{\partial(u\,w)}{\partial x}+\frac{\partial(v\,w)}{\partial y}+\frac{\partial(w^2)}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\Delta w$$

Let us replace u, v and p respectively by u + u ′ , v + v ′ and p + p ′ and then take the time average of the equations thus found.

the Navier–Stokes equations for steady-state turbulent flows:

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0$$
$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)=-\frac{\partial \bar{P}}{\partial x}+\mu\Delta\bar{u}-\rho\left(\frac{\partial\left(\overline{u'^2}\right)}{\partial x}+\frac{\partial\left(\overline{v'\,u'}\right)}{\partial y}+\frac{\partial\left(\overline{w'\,u'}\right)}{\partial z}\right)$$
$$\rho\left(\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}+\bar{w}\frac{\partial \bar{v}}{\partial z}\right)=-\frac{\partial \bar{P}}{\partial y}+\mu\Delta\bar{v}-\rho\left(\frac{\partial\left(\overline{u'\,v'}\right)}{\partial x}+\frac{\partial\left(\overline{v'^2}\right)}{\partial y}+\frac{\partial\overline{w'v'}}{\partial z}\right)$$
$$\rho\left(\bar{u}\frac{\partial \bar{w}}{\partial x}+\bar{v}\frac{\partial \bar{w}}{\partial y}+\bar{w}\frac{\partial \bar{w}}{\partial z}\right)=-\frac{\partial \bar{P}}{\partial z}+\mu\Delta\bar{w}-\rho\left(\frac{\partial\left(\overline{u'\,w'}\right)}{\partial x}+\frac{\partial\left(\overline{v'\,w'}\right)}{\partial y}+\frac{\partial\left(\overline{w'^2}\right)}{\partial z}\right)$$

At this stage a precise examination of these terms and a comparison with the stress tensor is necessary. It allows us to realize quite quickly that they represent components of the stress tensor due to the turbulent velocity:

$$\begin{pmatrix}\sigma'_{xx} & \sigma'_{xy} & \sigma'_{xz}\\ \sigma'_{xy} & \sigma'_{yy} & \sigma'_{yz}\\ \sigma'_{xz} & \sigma'_{yz} & \sigma'_{zz}\end{pmatrix}=\begin{pmatrix}\sigma'_{xx} & \tau'_{xy} & \tau'_{xz}\\ \tau'_{xy} & \sigma'_{yy} & \tau'_{yz}\\ \tau'_{xz} & \tau'_{yz} & \sigma'_{zz}\end{pmatrix}=-\rho\begin{pmatrix}\overline{u'^2} & \overline{v'\,u'} & \overline{w'\,u'}\\ \overline{u'\,v'} & \overline{v'^2} & \overline{w'v'}\\ \overline{u'\,w'} & \overline{v'\,w'} & \overline{w'^2}\end{pmatrix}$$

We rewrite the equations in the form:

$$\rho\left(\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\bar{w}\frac{\partial\bar{u}}{\partial z}\right)=-\frac{\partial\bar{P}}{\partial x}+\mu\Delta\bar{u}+\rho\left(\frac{\partial(\sigma'_{xx})}{\partial x}+\frac{\partial(\tau'_{xy})}{\partial y}+\frac{\partial(\tau'_{xz})}{\partial z}\right)$$

$$\rho\left(\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}+\bar{w}\frac{\partial\bar{v}}{\partial z}\right)=-\frac{\partial\bar{P}}{\partial y}+\mu\Delta\bar{v}+\rho\left(\frac{\partial(\tau'_{xy})}{\partial x}+\frac{\partial(\sigma'_{yy})}{\partial y}+\frac{\partial\tau'_{yz}}{\partial z}\right)$$

$$\rho\left(\bar{u}\frac{\partial\bar{w}}{\partial x}+\bar{v}\frac{\partial\bar{w}}{\partial y}+\bar{w}\frac{\partial\bar{w}}{\partial z}\right)=-\frac{\partial\bar{P}}{\partial z}+\mu\Delta\bar{w}+\rho\left(\frac{\partial(\tau'_{xz})}{\partial x}+\frac{\partial(\tau'_{yz})}{\partial y}+\frac{\partial(\sigma'_{zz})}{\partial z}\right)$$

In short, in index notations, the interpretation of the mean field and fluctuation can be illustrated as follows:

$$\frac{\partial\bar{u}_i}{\partial x_i}=0$$

$$\underbrace{\rho\frac{D\bar{u}_i}{Dt}}_{\substack{force\ moyenne\\ d'inertie}}=\underbrace{\frac{\partial\bar{P}}{\partial x_i}}_{\substack{force\ moyenne\ de\\ pression}}+\underbrace{\rho\bar{f}_i}_{\substack{force\ moyenn\\ e\ de\ volume}}+\underbrace{\mu\frac{\partial^2\bar{u}_j}{\partial x_i\partial x_j}}_{Force\ de\ viscosité}-\underbrace{\frac{\partial}{\partial x_j}\left(\rho\overline{u_i u_j}\right)}_{\substack{Tension\ de\\ Reynolds}}$$

5.6 Modelisation of turbulence

Direct Numerical Simulations (DNS)
Solving (numerically) the Navier-Stokes equations “directly”. without any turbulence models.

a. Very fine resolutions (both spatial and temporal) are required to resolve the motion of all eddies sufficiently.
b. Can be performed only for relatively low Re flows, i.e. cannot be used for most engineering flows.
c. Used mainly in academic research, to obtain the “exact” solutions of basic turbulent flows (this can be used for validation of turbulence models).
d. Also used to study detailed mechanisms of laminar-to-turbulent transition, combustion, bubble production, etc. • Sometimes called numerical experiments.

a. Reynolds Stress Models(RSM)
Modelling each of 6 Reynolds stress components induvidually, deriving and soliving a transport equation for each of the 6 components. Required relatively high computational cost**.**

b. Eddy viscosity models(EVM)
Modelling each of 6 Reynolds stress components all together, assuming a similarity between the viscous stress(due to viscosity) and Reynolds stress(due to turbulence). Requires relatively low computational cost.

5.7 Eddy viscosity models(EVM)

Eddy viscosity models are often classified into the following 3 groups, depending on how many “model transport equations” are solved to estimate v_t.

Zero- equation (or algebric) models

a. Prandtl’s mixing length model(1925);
b. Cebeci- Smith model(1967);
c. Baldwin-Lomax model(1978) …ect

One- Equation models

a. Prandtl’s one equation model(1945)
b. Baldwin Barth model(1990)
c. Spalart allmaras model(1994) ..ect

Two equation models(most commonly used)

a. $k - \varepsilon$ models(Lunder & Spalding 1974)
b. $k - \omega$ models(Wilcox 1988, 1993, 1998)
c. Hybrid models ($k - \omega\ SST$ model by Menter 1994)

Chapter 6: Calculation of flow in pipes

6.1 Pressure losses: application to real fluids

The head loss in Bernoulli's equation represents the reduction in the total pressure, which is the sum of the velocity head, pressure head, and the elevation head of the fluid flowing through the hydraulic system.

There are two types of pressure losses, linear pressure losses related to length and singular losses related to changes in the shape of the fluid flow circuit (variation of the fluid passage section).

6.2 Singular pressure losses

This type of pressure loss is linked to specific accidents (change in the shape of pipes), for example sudden or gradual widening and narrowing, bends, shapes (U, T and Y) valves, taps, flaps, etc..

They are expressed) from a dimensionless load loss coefficient noted by:

"$\boldsymbol{J_S}$" and given by the following formula:

$$J_S = \xi \frac{v^2}{2g}$$

with :

ξ : Coefficient of singular load losses,

v: average flow velocity at the passage section considered,

g : gravitational acceleration

The coefficient ξ is determined explicitly for some cases (or by empirical formulas), or incurred by abacuses. We can retain some examples of the cases below:

Abrupt widening: $\boldsymbol{\xi = \left(1 - \frac{S_1}{S_2}\right)^2}$

Abrupt contraction: $\boldsymbol{\xi = 0.45\left(1 - \frac{S_2}{S_1}\right)^2}$

Elbow 90° : $\boldsymbol{\xi}$ **= 0.8**

Open valve: $\boldsymbol{\xi}$ **= 1.2**

6.3 Linear or friction pressure losses for the different flow types: (Formula of Colebrook: Moody diagram).

This type of loss is caused by the internal friction that occurs in liquids; it occurs in smooth pipes as well as in rough pipes.

Between two points separated by a length L, in a pipe of diameter D there appears a pressure loss p. expressed in the following form:

Let us use Poiseuille's result to express the pressure loss by involving the diameter D, the length L, the average speed as well as the density and the Reynolds number:

$$\mathbf{Q} = \frac{\Delta \mathrm{Pr}\,\pi}{8\mu L}\mathbf{R^4} \Rightarrow \mathbf{\Delta Pr} = \frac{Q\,8\,\mu\,L}{\pi R^4} = \frac{v\,\pi\,R^2\,8\,\mu\,L}{\pi R^4} = \frac{v8\mu L}{R^2} = \frac{v8\mu L}{\frac{D^2}{2}} = \frac{32 v\mu L}{D^2}$$

The Reynolds number of this laminar flow is written as: $\boldsymbol{Re} = \frac{\rho\,V\,D}{\mu}$

so

$$\Rightarrow \mathbf{\Delta Pr} = \frac{1}{2}\frac{64\ L}{\frac{\rho\,V\,D}{\mu}}\ \rho\,V^2\ \frac{L}{D} = \frac{64}{Re}\ \frac{1}{2}\ \rho\,V^2\ \frac{L}{D} = \ \lambda\ \frac{1}{2}\ \rho\,V^2\ \frac{L}{D}\quad (\mathbf{Pa})$$

(Formula of Darcy-Weisbach)

with: $\boldsymbol{\lambda}$=$\mathbf{64}/\boldsymbol{Re}$

We define the linear pressure loss:

$\mathbf{\Delta H} = \frac{\Delta Pr}{\rho\,g} = \ \lambda\ \frac{1}{2g}\ \ V^2\ \frac{L}{D}$ (**Pa**) *Pressure loss expressed in meters of fluid column (mFC)*

λ is a dimensionless coefficient called the linear pressure loss coefficient.

The calculation of pressure losses is based entirely on the determination of this coefficient.

Laminar flow case : Re < 2000

In this case we note that the pressure loss coefficient is only a function of the Reynolds number Re; the state of the surface does not intervene and therefore does not depend on the nature of the piping.

$\boldsymbol{\lambda} = \frac{64}{Re}$ (Poiseuille law)

With , $\boldsymbol{Re} = \frac{\rho\,V\,D}{\mu}$

Flow case 2000 < Re < 105

In this case, the turbulence is still moderate, the pipe surface is still considered smooth. Here again λ only depends on Re

$\boldsymbol{\lambda}$**= (100** $\boldsymbol{Re}$**)**$^{-0.25}$ **(Formule de Blasius)**

$\frac{1}{\sqrt{\lambda}} = \mathbf{2}\ \boldsymbol{log}\left(\frac{Re\,\sqrt{\lambda}}{2{,}51}\right)$ **(Von Karman implicit formula)**

Flow case: Re > 10^5

The turbulence becomes very significant and λ only depends on ε/D

$\frac{\mathbf{1}}{\sqrt{\boldsymbol{\lambda}}} = \mathbf{2}\ \boldsymbol{log}\left(\mathbf{3,71}\frac{\boldsymbol{\varepsilon}}{\boldsymbol{D}}\right)$ **(Formula of Nikuradse)**

with, ε average height of the asperities of the pipe surface or absolute roughness, ε/D étant la relative roughness, and D section diameter.

Note:

The Colebrook formula is currently considered to be the one that best reflects flow phenomena in turbulent conditions. It is presented in the following form:

$\frac{\mathbf{1}}{\sqrt{\boldsymbol{\lambda}}} = -\mathbf{2}\ \boldsymbol{log}\left(\frac{\boldsymbol{\varepsilon}}{\mathbf{3.7}\ \boldsymbol{D}} + \frac{\mathbf{2,51}}{\boldsymbol{Re}\ \sqrt{\boldsymbol{\lambda}}}\right)$ **Formula of Colebrook**

6.6 Moody diagram.

In 1944,Lewis Ferry Moody plotted theDarcy- Weisbach friction factor againt Reynolds number Re for various values of roughness $^{\varepsilon}/_{D}$

The Moody friction factor $\boldsymbol{\lambda}$ is used in the Darcy-Weisbach major loss equation. The coefficient can be estimated with the diagram below:

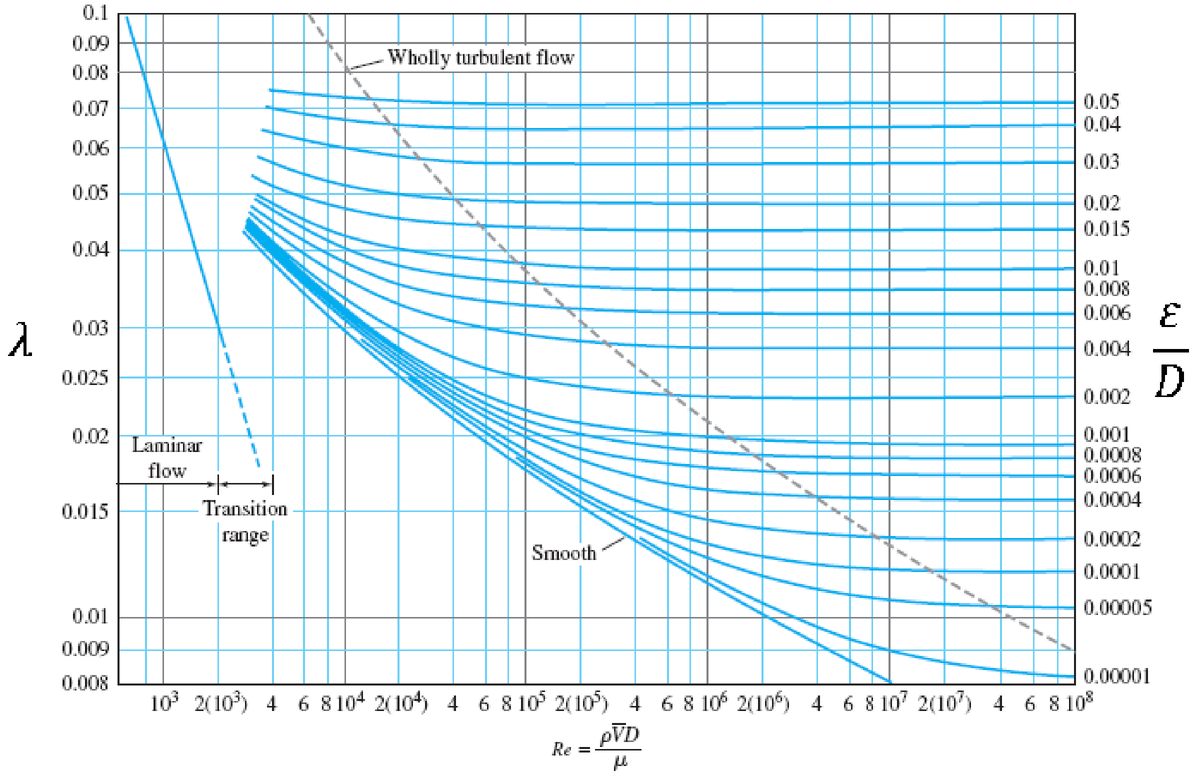


Fig 6.Moody diagram

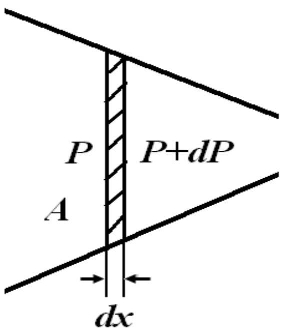
6.7 Introduction to one-dimensional flow of compressible fluids

If the flow is transient - 2300 < Re < 4000 - the flow varies between laminar and turbulent flow and the friction coefiicient is not possible to determine. The friction factor can usually be interpolated between the laminar value at Re = 2300 and the turbulent value at Re = 4000

6.7.1 Fundamental equations

6.7.1.1 Mass conservation equation

$$\boldsymbol{\dot{m} = C.\rho.A} \qquad \left({}^{\boldsymbol{kg}}/_{\boldsymbol{s}}\right)$$



$\dot{m}$: Mass flow rate through a given section A, for steady flow.

$\dot{m} = cst$, $\dot{m} = \frac{A.C}{v}$, $\dot{m}\,.v = A.C$

$\frac{\dot{m}}{A} = \rho\,.C = g_m$: Specific mass flow rate.

If A=cst so $g_m = cst \quad \rho.C = cst$

6.7.1. 2 Equation of motion

Following Newton's principle

$$F = \underbrace{m}_{masse} \;.\; \underbrace{\frac{dC}{dt}}_{accélération}$$

$$\underbrace{(P - P - dP).A}_{force} = \underbrace{\rho.A.dx}_{masse}. \; \underbrace{\frac{dc}{dt}}_{accélération}$$

With : $\frac{dx}{dt} = C$ we will have: $-dP = \rho.d\left(\frac{C^2}{2}\right)$

Either :

$$-v\,dP = d\left(\frac{C^2}{2}\right)$$

6.7.1.3 Energy equation

According to the first law of thermodynamics

$dq = dh - v\,dP$ Mais $-v\,dP = d\left(\frac{C^2}{2}\right)$

$\Rightarrow d\left(\frac{C^2}{2}\right) + dh = 0$ Or as an integral $\frac{C^2}{2} + h = cst$: equation of conservation of energy.

We know that $h = c_p.T$ and $c_p = \frac{\gamma}{\gamma-1}\,r$

$$\Rightarrow \frac{C^2}{2} + \frac{\gamma}{\gamma-1}\,r.T = cte$$

$$\Rightarrow \frac{C^2}{2} + \frac{\gamma}{\gamma-1}\frac{P}{\rho} = cte$$

6.7.2 Stagnation parameters and generating parameters

6.7.2.1 generating state

The characteristics inside this reservoir are then those of the generating state, this representation justifies the name “generating state”, it is characterized by the index i.

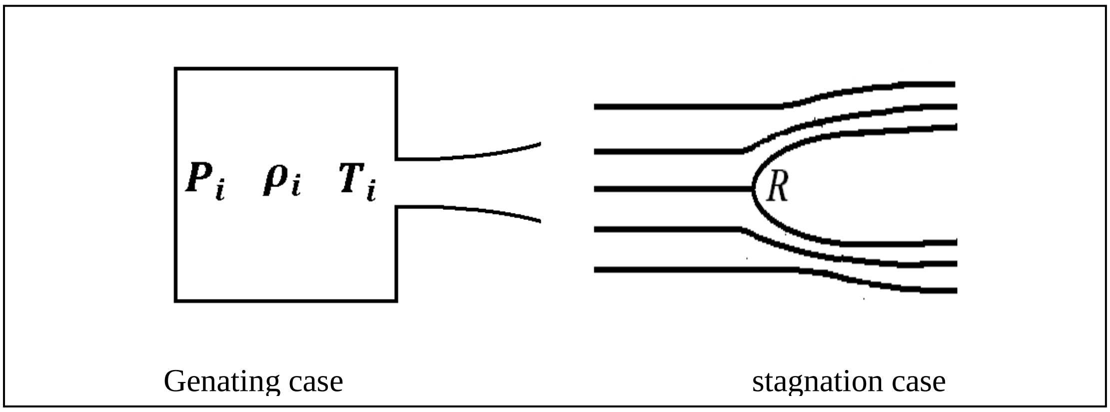


Fig 6.3 Stagnation parameters and generating parameters

6.7.2.2 Stagnation state

Let a breakpoint be defined in the following figure:

$$\frac{C^2}{2} + h = cst \; = h_0 = h_i = c_p T_0$$

$$\Rightarrow T_0 = T + \frac{C^2}{2c_p} \qquad \rightarrow \text{Stagnation temperature}$$

This is the temperature that will be measured with a thermometer fixed in the flow.

as $h_0 =$ cte donc $T_0 = cst$ (identical in each section of the flow).

h_0 : Stagnation enthalpy.

$$\frac{T_0}{T} = 1 + \frac{C^2}{2h} \quad \text{and} \qquad h = c_p.T = \frac{\gamma}{\gamma - 1}\, r \frac{a^2}{\gamma\, r} = \frac{a}{\gamma - 1}$$

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

$$\frac{P_0}{P} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/\gamma - 1}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/\gamma - 1}$$

In case the gas flows from the medium where the initial velocity C_0 is negligible compared to the speed in the section considered C, we have :

$$\frac{C^2}{2} - \frac{C_i^2}{2} = h_i - h \quad \Rightarrow C = \sqrt{2(h_i - h)}$$

h_i : Enthalpie génératrice, $h_i = c_p\, T_i$ avec T_i : generating temperature.

$T_i = \frac{P_i}{\rho_i\, .r}$ ou P_i : generating pressure, ρ_i : generating density.

- As $\frac{C^2}{2} + h = h_i$, we note that $h_i = h_0$ et $T_i = T_0$
- - In an adiabatic flow the stopping enthalpy and the stagnation temperature are identified with the generating enthalpy and temperature, consequently:
  $P_i = P_0$ et $\rho_i = \rho_0$, so we write:

$$\frac{T_i}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

$$\frac{P_i}{P} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/_{\gamma-1}}$$

$$\frac{\rho_i}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/_{\gamma-1}}$$

6.7.3 Comparison with incompressible flow

$$\boldsymbol{P_0 - P = \rho\ \frac{C^2}{2} \qquad \Rightarrow \qquad \frac{P_0 - P}{\rho\ C^2/_2} = 1}$$

For a compressible fluid in subsonic isentropic flow:

$$\frac{P_i}{P} = \left(1 + \frac{(\gamma - 1)\ M^2}{2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{P_i}{P} = 1 + \frac{\gamma}{2} M^2 + \frac{\gamma}{8} M^4 + \frac{\gamma(2-\gamma)}{48} M^6 + \ldots\ldots\ldots$$

$$P_i - P = \frac{P\gamma\ M^2}{2}\left(1 + \frac{M^2}{4} + \frac{2-\gamma}{24} M^4\right)$$

$$\frac{P\gamma\ M^2}{2} = \frac{P\gamma\ C^2}{2\ \gamma\ rT} = \frac{\rho\ C^2}{2}$$

where :

$$\frac{P_i - P}{\rho\ C^2/_2} = 1 + \frac{M^2}{4} + \frac{M^4}{40} + \frac{M^6}{1600} + \ldots..$$

Hence the following table:

M	0.1	0.2	0.3	0.4	0.5
$\frac{P_i - P}{\rho\ C^2/_2}$	1.003	1.010	1.023	1.041	1.064

At low velocity when $\frac{M^2}{4} < 1$ Bernoulli's formula is modified:

$$\frac{P_i - P}{\rho\, C^2 /_2} = 1$$

$$P_i - P \quad = {}^{\rho\, C^2}/_2 \quad \text{et} \quad P_0 - P \quad = {}^{\rho\, C^2}/_2$$

$\Rightarrow P_0 = P_i = P \quad + {}^{\rho\, C^2}/_2$ Bernoulli's formula for incompressible fluid.

${}^{\rho\, C^2}/_2$: Dynamic pressure

As $\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{1}{\gamma-1}}$ si $\frac{\gamma-1}{2} M^2 < 1$

$\frac{\rho_0}{\rho} \approx 1 + \frac{M^2}{2} + \cdots$ The relative variation in density between the upstream state and any section.

$$\frac{\rho_0}{\rho} - 1 = \frac{\rho_0 - \rho}{\rho} = \frac{M^2}{2} = \frac{\rho_i - \rho}{\rho}$$

for :

M=0.1 $\quad \frac{\rho_0 - \rho}{\rho} = 0.5\%$

M=0.14 $\quad \frac{\rho_0 - \rho}{\rho} = 1\%$

M=0.2 $\quad \frac{\rho_0 - \rho}{\rho} = 2\%$

6.7.4 Formula of Barré- saint venant

Given the results concerning the velocity calculation in the form,

$$C = \sqrt{2(h_0 - h \quad)}$$

We can give it an expression called « formula of Barré-saint venant »

$$C = \sqrt{2\, cp\, (T_0 - T\ )} = \sqrt{2\, cpT_0 \left(1 - \frac{T}{T_0}\right)}$$

$$C\ = \sqrt{2\ \frac{\gamma}{\gamma - 1}}$$

$C = \sqrt{\frac{2\gamma}{\gamma-1}\frac{P_0}{\rho}(1 - \left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}})}$ ***<u>Formula of Barré- saint venant</u>***

6.7.5 Critical parameters

When gas expansion takes place in a pipeline, the sonic state separates regions of flow that are fully accessible (subsonic) and those that are not (supersonic)

$$M = 1 \Rightarrow \frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

$$\Rightarrow \frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}(1)^2$$

$$\Rightarrow T = T_C = \frac{2}{\gamma + 1} T_0$$

With M=1 gives us the so-called critical temperature.

$$\frac{P_C}{P} = \left(\frac{2}{\gamma + 1}\right)^{\gamma/_{\gamma-1}}$$

For the critical section

$$C_c\ = a_c\ = \sqrt{\gamma\ \ r\ \ T_c}\ = \sqrt{\frac{2\,\gamma}{\gamma+1} r\ \ T_0}\ = \sqrt{\frac{2\,\gamma}{\gamma+1}\frac{P_0}{\rho_0}}$$

The critical speed is a function of the nature of the gas and the generating (or stagnation state) conditions.As $a_0 = \sqrt{\gamma\ \ r\ \ T_0}$, we note that:

$$C_c = a_c = \sqrt{\frac{2\,\gamma}{\gamma+1}\frac{P_0}{\rho_0}} = \sqrt{\frac{2 \quad \gamma\, r\, T_0}{\gamma+1}} = a_0\sqrt{\frac{2}{\gamma+1}}$$

6.8 Introduction to free surface flow

Free surface flows are flows that flow under the effect of gravity while being in partial contact with a container (canal, river, pipe) and with air whose pressure is generally at free surface.

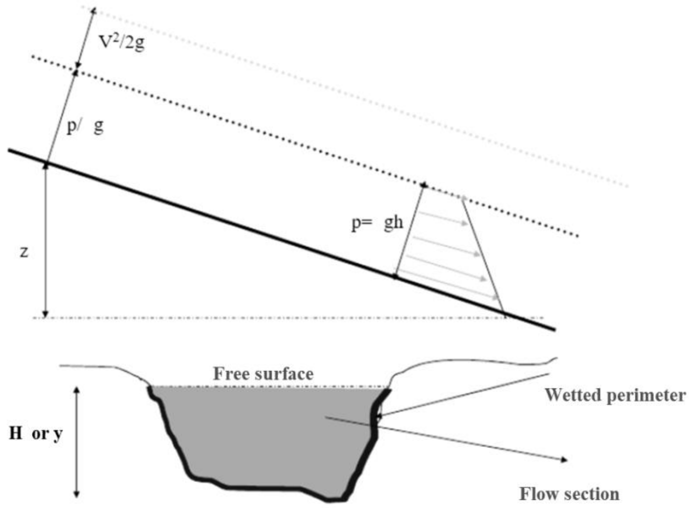


Fig 6.4 free surface flow

Type of flow: A classification of flows can be made according to the variation of depth, h or dh, as a function of time and space: dh = f(t, x).

The types of flow encountered in free surface hydraulics can be summarized as follows:
1. Uniform flow
2. Non-uniform flow: gradually varied and abruptly varied

6.8.1 Flow regimes

The Froude number, which is the ratio of gravity to inertial forces or:

$$\boldsymbol{Fr = \frac{V}{\sqrt{g.h}}}$$

river flow: $\boldsymbol{Fr < 1}$ torrential flow: $\boldsymbol{Fr > 1}$ critical flow: $\boldsymbol{Fr} = \boldsymbol{Fr_{cr}}$
Uniform regime:
- The flow is considered uniform if the height h is constant along the flow.

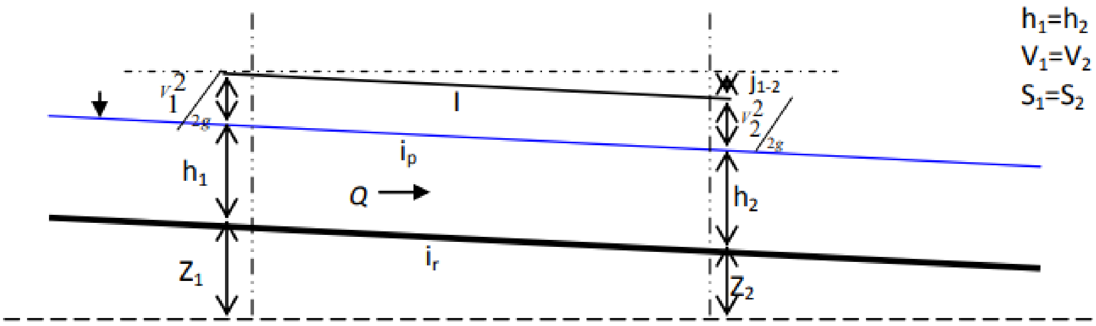


Fig 7.5 load plan of Uniform flow

$$S_{wet} = cst, \qquad P_{wet} = cst, R_h = \frac{S_{wet}}{P_{wet}}, \qquad h = cst$$

Uniform flow must satisfy the following conditions:

1. The flow rate of the water is constant.
2. The channel is prismatic.
3. The slope of the invert is constant.
4. The roughness of the channel is constant along the flow.
5. The lines of the flow are parallel.

6.8.2 Main calculation formulas

1- Mean flow velocity: The main calculation formula for free surface flow is that of Chezy:

Where: V: flow velocity. C: Chezy coefficient. R: hydraulic radius. i: the slope of the channel bottom (slope of the invert) To determine C, one of the following formulas can be used:

formula of Bazin: $C = \frac{87\sqrt{R}}{\Gamma + \sqrt{R}}$

Formula of Kutter: $C = \frac{100\sqrt{R}}{\varepsilon + \sqrt{R}}$

Formula of Maning: $C = \frac{1}{n} R^{1/6}$

Γ, ε and n are coefficients that depend on the roughness of the walls

$\Gamma = 0.16,\ \varepsilon = 0.2,\ n = 0.0125 \Rightarrow brick, stone\ or\ board\ channels$

The flow rate:

$$Q = V \times S = C.S\sqrt{R.i}$$

6.8.3 Non-uniform regime (varied permanent)

Definition: When the trajectories of the different liquid streams flowing in a channel are not parallel to each other, we have a varied regime (the free surface and the bottom of the channel are not parallel). This type of movement occurs in a channel in the varied cross section (like natural watercourses)

Bernoulli's theorem: We apply Bernoulli's equation between the sections 1.1 et 2.2

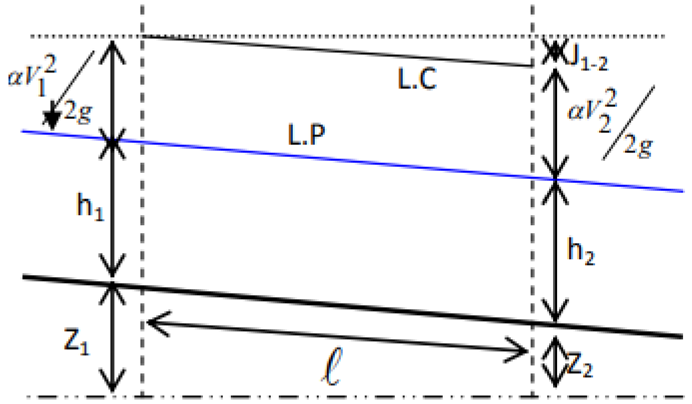


Fig 7.5 load plan of non- uniform flow

$$\boldsymbol{H_1 + \frac{\alpha_1 V_1^2}{2g} = H_2 + \frac{\alpha_2 V_2^2}{2g} + \Delta H}$$

Exercises related to chapter 1

EXERCISE

Find the streamlines and equipotentials in the following cases:

The complex potential is: f(z) = V.z where V is a real constant and Z = x +i y

The complex potential is: f(z) = k ln(z) where k is a real constant

<u>Exercice 1</u>

We consider a permanent flow defined in a frame (0, x, y, z) by the following velocity field, in Euler variables:

$$\vec{V}\begin{cases} u = 2x - 3z \\ v = 0 \\ w = 3x - 2z \end{cases}$$

1) Show that the fluid is incompressible.

2) Calculate the acceleration vector field.

3) Determine the equations of the streamline network.

4) Determine the strain rate tensor field..

Solution

$$\vec{a} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}.\vec{\nabla})\vec{V}$$

The streamlines are defined by the equation:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

1) Show that the fluid is incompressible.

We must show that $div\vec{V} = 0$.

We just need to check that the following equation is true:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

After a quick calculation we get:

$$\frac{\partial u}{\partial x} = 2\ ;\ \frac{\partial v}{\partial y} = 0\ ;\ \frac{\partial w}{\partial z} = -2$$

The sum of these 3 terms is zero, the fluid is indeed incompressible.

2) Calculate the acceleration vector field

Acceleration is defined by:

$$\vec{a} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}.\vec{\nabla})\vec{V}$$

The flow is permanent hence

$$\frac{\partial \vec{V}}{\partial t} = 0 \text{ so } \vec{a} = (\vec{V}.\overrightarrow{grad})\vec{V}$$

After calculation we obtain:

$$\vec{a}\begin{cases} (2x-3z)\dfrac{\partial}{\partial x}(2x-3z)+(3x-2z)\dfrac{\partial}{\partial z}(2x-3z) \\ 0 \\ (2x-3z)\dfrac{\partial}{\partial x}(3x-2z)+(3x-2z)\dfrac{\partial}{\partial z}(3x-2z) \end{cases}$$

$$\text{D'où } \vec{a}\begin{cases} 2.(2x-3z)+(3x-2z)(-3) \\ 0 \\ (2x-3z).3+(3x-2z)(-2) \end{cases} \quad \text{et donc } \boxed{\vec{a}\begin{cases} -5x \\ 0 \\ -5z \end{cases}}$$

2) Calculate the acceleration vector field

Acceleration is defined by:

$$\vec{a}=\frac{\partial\vec{V}}{\partial t}+(\vec{V}.\vec{\nabla})\vec{V}$$

The flow is permanent hence $\dfrac{\partial\vec{V}}{\partial t}=0$ so $\vec{a}=(\vec{V}.\overrightarrow{grad})\vec{V}$

The flow is permanent hence

$$\vec{a}\begin{cases} (2x-3z)\dfrac{\partial}{\partial x}(2x-3z)+(3x-2z)\dfrac{\partial}{\partial z}(2x-3z) \\ 0 \\ (2x-3z)\dfrac{\partial}{\partial x}(3x-2z)+(3x-2z)\dfrac{\partial}{\partial z}(3x-2z) \end{cases}$$

$$\text{D'où } \vec{a}\begin{cases} 2.(2x-3z)+(3x-2z)(-3) \\ 0 \\ (2x-3z).3+(3x-2z)(-2) \end{cases} \quad \text{et donc } \boxed{\vec{a}\begin{cases} -5x \\ 0 \\ -5z \end{cases}}$$

3) Determine the equations of the streamline

The streamlines are defined by

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

We have v=0. For this equation to be defined, dy=0 must be present. The flow is in the Oxz plane.

We were going to transform this equation...

$$\frac{dx}{2x-3z}=\frac{dz}{3x-2z}$$

What gives us $(3x-2z).dx=(2x-3z).dz$

Then

$3x.dx-2d.(xz)+3zdz=0$

Finally

$$\boxed{\frac{3}{2}x^2+\frac{3}{2}z^2-2xz=cst}$$

4) Determine the strain rate tensor field.

By definition the strain rate tensor is given by***:***

$$\overline{\overline{D}}=\frac{1}{2}\begin{bmatrix}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} & \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\\ \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}+\frac{\partial w}{\partial z}\end{bmatrix}$$

After simplification and a quick calculation, we obtain:

$$\overline{\overline{D}}=\frac{1}{2}\begin{bmatrix}4 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -4\end{bmatrix}\longrightarrow\boxed{\overline{\overline{D}}=\begin{bmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -2\end{bmatrix}}$$

<u>Exercice 2</u>

Find the streamlines and equipotentials in the following cases:

The complex potential is: : $f(z)=V.z$ where V is a real constant and $\mathrm{z}=\mathrm{x}+\mathrm{iy}$.

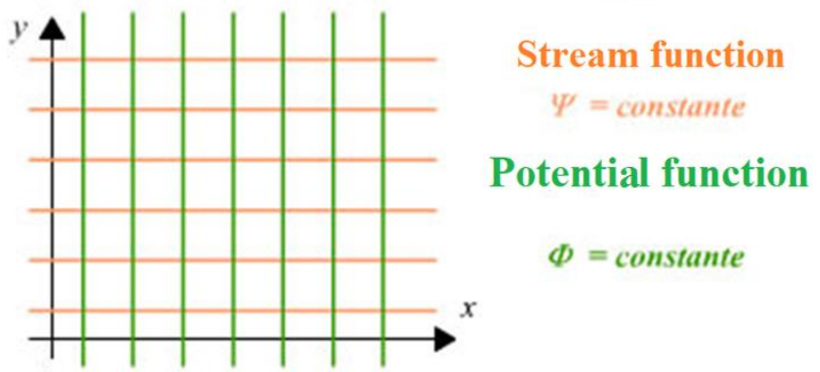
<u>Solution</u>

So $f(z)=V.(\mathrm{x}+\mathrm{iy})$

We have more $f(z)=\Phi+i\Psi$ avec Φ the potential functionet Ψ the stream function.

We therefore obtain $\Phi=\mathrm{V.x}$ and $\Psi=\mathrm{V.y}$

The streamlines are given for $\Psi=cst$ and the equipotentials are given for $\Phi=\mathrm{cst}$



Exercise 3

Find the streamlines and equipotentials in the following cases:

The complex potential is: $f(z) = k.\ln(z)$ where k is a real constant.

Solution

Find the streamlines and equipotentials in the following case:

The complex potential is: $f(z) = k.\ln(z)$ where k is a real constant.

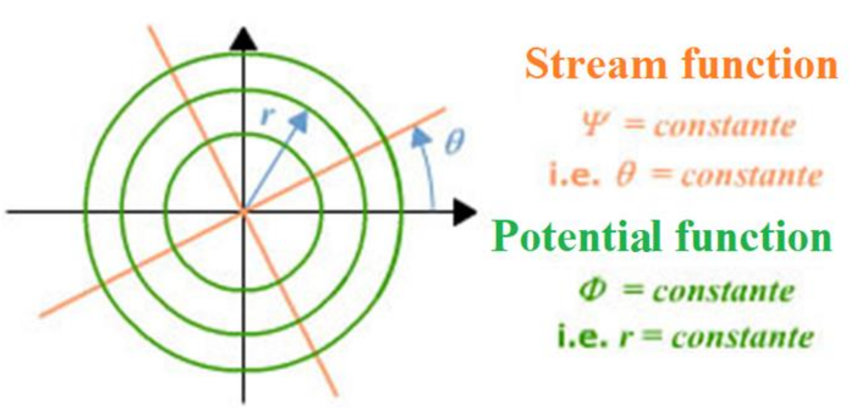
Using the polar coordinates, we have $f(z) = k.\ln(r.e^{i\theta})$

From where $f(z) = k.ln\, r + ki\ \theta$

Moreover $f(z) = \Phi + i\Psi$ with Φ the potential function et Ψla fonction courant.

We therefore obtain $\Phi = \mathrm{k.lnr}$ and $\Psi = k.\theta$

The streamlines are given for $\Psi = cst$ and the equipotentials are given for $\Phi = \mathrm{cst}$

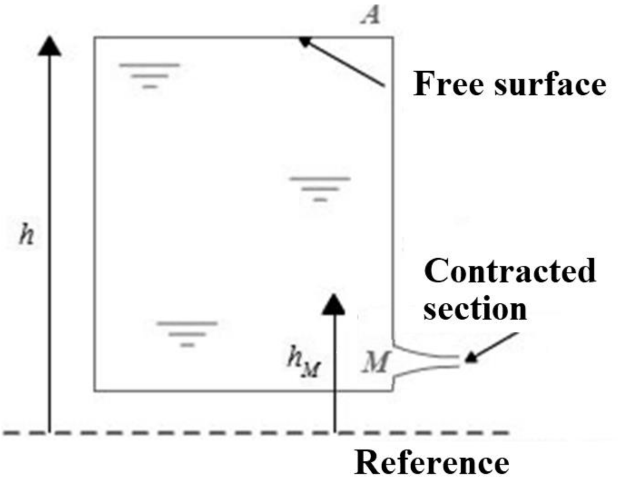


Exercises related to chapter 2

Exercise 1 (Draining a reservoir)

Consider a reservoir whose fluid escapes through a narrow orifice. The tank is large enough to neglect the variations in the level of the free surface over time and to consider the permanent motion. The fluid is considered perfect.

At the outlet, the jet has a contracted section where the streamlines are parallel and practically rectilinear.



Question

1 - Calculate the velocity at point M at the outlet of the orifice.

2 - Calculate the draining time.

Solution

1 - Calculate the velocity at point M at the outlet of the orifice

There is a streamline between point A located on the free surface and point M in the outlet section, so we can apply the Bernoulli relation between these two points:

$$P_A + \rho\, gh_A + \rho \frac{V^2 A}{2} = P_M + \rho\, gh_M + \rho \frac{V^2 A}{2}$$

Considering the flow conditions, we have: $P_A = P_M = P_{atm}$. Furthermore, since the section of the reservoir is large compared to that of the orifice, the velocity at A is negligible compared to that of M: VA = 0 (it is sufficient to apply the conservation of flow rate to realize this). By integrating these data into the equation, we obtain:

$$V = \sqrt{2\, g(h_A - h_M)} \qquad h_A - h_M = z$$

We obtaine the Torricelli formula: $\boxed{V = \sqrt{2\, g\, z}}$

2 - Calculate the emptying time

We express that the volume evacuated during the time is equal to the decrease in volume in the reservoir, so:

The volume evacuated during the time Δt is equal to: $q_v dt$ where q_v is the volume flow rate.

The decrease in volume in the reservoir is: $-S\,dh$ where S is the section of the reservoir and dh the height variation in the reservoir over time Δt

The volume flow rate is equal to: $\boldsymbol{q_v} = C_c V S_0$. In this expression This is the contraction coefficient (the section of the jet at the outlet is $C_c S_0$ and V the fluid velocity at the orifice. Equality is written:

$c_c S_0 V\ dt = -S\ dt$

We obtain: $dt = -\frac{Sdz}{c_c S_0 V}$

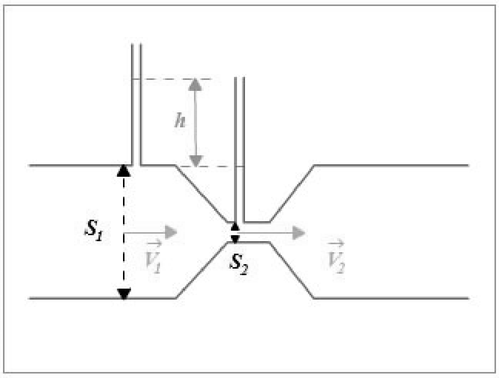
Either by replacing V with its value: V

$$dt = -\frac{S}{c_c S_0 \sqrt{2g}} \frac{dz}{\sqrt{z}}$$

For a complete drain, the integration between h_A and zero gives:

$$\boxed{t = \frac{S}{c_c S_0 \sqrt{2g}} \int_{h_A}^{0} \frac{dz}{\sqrt{z}} = \frac{S}{c_c S_0 \sqrt{2g}} \sqrt{h_A}}$$

Exercise 2 (Venturi tube)
The venturi tube is a convergent-divergent tube equipped with static pressure tapping, one upstream of the convergent, the other at the neck (see figure).



This tube is inserted into a pipe whose flow rate is to be measured. Water (incompressible perfect fluid) flows into the venturi and h is the difference in level in the tubes indicating the pressure. The speeds in S_1 *and* S_2 are uniform.

Question

1 - Calculate the fluid velocity in the contracted section as a function of the sections and and the pressure difference at and at .

2 - Express the flow rate (in volume) of the pipe.

1 - Calculate velocityV_2 of the fluid in the contracted section as a function of the S_1 and S_2 of the difference in pressures P_1atS_1 and P_2atS_2 .

2 - Express the flow rate (in volume) of the pipe

Solution

1 - Calculate the velocity

P_1 and $P_2$the pressures in the sections and . We apply the Bernoulli relation:

$$P_1 + \rho g z_1 + \rho \frac{V_1^2}{2} = P_2 + \rho g z_2 + \rho \frac{V_2^2}{2}$$

z_1 and z_2 are the respective coasts of the chosen streamline passing through the sections S_1 and S_2.

Let's call P_1^*let P_2^* the terms $P + \rho g z$. with the conservation of volume flow $(V_1 S_1 = V_2 S_2)$, we obtain:

$$\boxed{V_2 = \sqrt{\frac{2g}{1-(\frac{S_2}{S_1})^2}}\sqrt{\frac{P_1^* - P_2^*}{\rho g}}}$$

2 - Express the flow rate in volume of the pipe.

The flow rate of the pipe is given by the following formula:

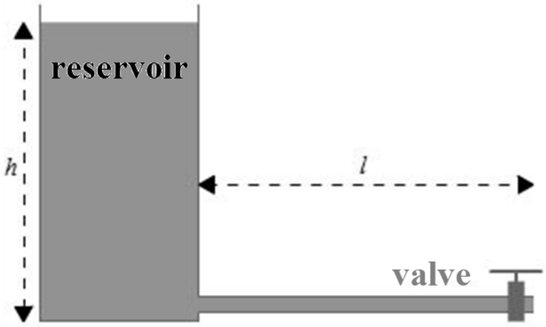
$$q_v = V_2 S_2 = S_2 \sqrt{\frac{2g}{1-(\frac{S_2}{S_1})^2}}\sqrt{\frac{P_1^* - P_2^*}{\rho g}}$$

Which gives us with $P_1^* - P_2^* = \rho g h$

$$\boxed{q_v = S_2 \sqrt{\frac{2gh}{1-(\frac{S_2}{S_1})^2}}}$$

Exercice 3

We consider a horizontal pipe, of constant section, of length l, supplied by a large reservoir where the level is kept constant. At the end of the pipe, a valve regulates the flow. At time t = 0, the valve is closed and is opened abruptly.



Question

Establish the relationship between the flow establishment time and the maximum fluid velocity.

<u>Solution</u>

Establish the formula between the flow establishment time and the maximum fluid velocity.

At a point at distance x from O, the Bernoulli formula in non-steady state is written:

$$P_a + \rho g h = P + \rho\frac{V^2}{2} + \rho\frac{\partial V}{\partial t}x$$

La section du tuyau est constante donc V et $\partial V/_{\partial t}$ont la même valeur le long du tuyau. En $x = l, P = P_a$, la relation précédente s'écrit donc :

$$\frac{\partial V}{\partial t}\frac{g}{l}(h - \frac{V^2}{2g})$$

Since V only depends on time, we can write$\partial V/_{\partial t} = {dV}/_{dt}$. The equation therefore becomes:

$$dt = 2l\frac{dV}{2gh - V^2}$$

By integrating, we obtain:

$$t = \frac{l}{\sqrt{2gh}}ln(\frac{\sqrt{2gh} + V}{\sqrt{2gh} - V})$$

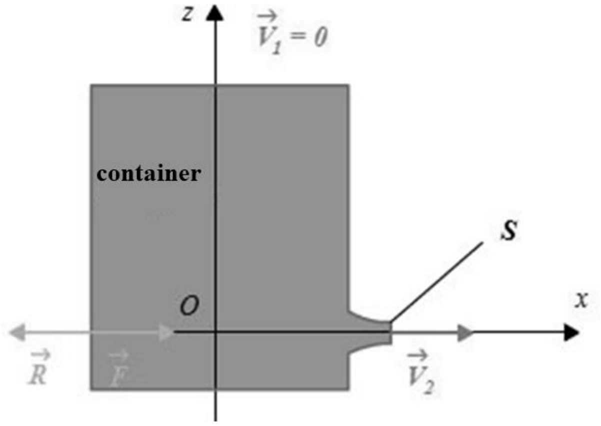
The previous integration shows a constant, but this is zero because the speed is zero at t=0.

When $t \to \infty$; $V = V_{max} = \sqrt{2gh}$, we are in the case of permanent flow (Torricelli formula), we can therefore write:

$$\boxed{t = \frac{l}{V_{max}}ln(\frac{1 + \frac{V}{V_{max}}}{1 - \frac{V}{V_{max}}})}$$

Exercise 4 (Reaction of a jet)

Consider a large container, pierced with an orifice from which a horizontal jet escapes. Apply the theorem of quantities of movement to the flow tube (the reference surface) limited by the free surface of the liquid, the container and the jet up to the contracted section S.



Calculate the thrust due to the fluid on the reservoir.

Appling Euler's theorem: $q_m(\overrightarrow{V_2} - \overrightarrow{V_1}) = \overrightarrow{P} + \overrightarrow{F}$

Let's project this relationship onto the coordinate axes:

On the Ox axis, the weight does not intervene:

$F = q_m V_2$avec $q_m = \rho S V_2$

So:

$$F = \rho S V_2^2$$

F is the resultant of the pressure forces exerted from the outside on the surface. Conversely, the tank experiences an equal and opposite thrust called jet reaction:

$R = -\rho S V_2^2$ And $(V_2^2 = 2gz)$

We obtain:

$$\boxed{R = -2\rho gz}$$

Exercises related to chapter 3

Problem 1

We consider an incompressible fluid of dynamic viscosity μ, of density ρ above a flat plate of infinite extent. This plate performs an oscillatory motion in its own plane. Because of the viscosity of the fluid, longitudinal oscillations are generated in the fluid above the plate.

In this case, the velocity of a point M of the fluid has only one component u along the x axis and this component depends only on z and time t (u = u(z,t)) where z is the vertical axis in an inertial frame.
The boundary conditions are as follows:
for z = 0, u(0,t) = Uexp(iωt) where U is the amplitude of the velocity of the plate and ω the pulsation of the oscillations for $z \to \infty, u \to 0$
1 - Determine the differential equation obeyed by the speed u(z,t). We will set ν = μ/ρ where ν is the kinematic viscosity.

2 - Find the solution to this equation. To do this, we will proceed by the method of separation of variables by setting:

U(z,t) = f(t).g(z)

where f(t) is a function depending only on t and g(z) a function depending only on z.

solution
1 - Determine the differential equation obeyed by the speed u(z,t).

We write the Navier Stokes equation in the present case:

$$\rho\frac{d\vec{V}}{dt} = -\overrightarrow{grad}P + \mu\Delta u$$

After some calculations we obtain:

$$\rho\left(\frac{\partial\vec{V}}{\partial t} + \vec{V}\overrightarrow{grad}\vec{V}\right)\frac{d\vec{V}}{dt} = -\overrightarrow{grad}P + \mu\Delta u$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

The terms in color are zero:

u only depends on z and t.

The motion is oscillatory along x.

V has no vertical component.

We therefore obtain:

$$\boxed{\rho\left(\frac{\partial u}{\partial t}\right) = \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial z^2}\right) = \nu\left(\frac{\partial^2 u}{\partial z^2}\right)}$$

2 2 - Find the solution to this equation

We set U(z,t) = f(t).g(z). We can then write our equation:
Calculate the characteristic distance δ for which the amplitude is damped to 1/e of its value at z = 0.

$$\left(\frac{i\omega}{2\nu}\right)^{1/2} = \left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}} \exp \quad \left(i\,\frac{\pi}{2\nu}\right)^{\frac{1}{2}} (1 + \mathrm{i})$$

We then obtain:

$$u(z,t) = U_{exp}\left(-\left(\frac{\omega}{2\nu}\right)^{1/2} z + i(\omega t - z\left(\left(\frac{\omega}{2\nu}\right)^{1/2}\right)\right)$$

We therefore have an oscillatory movement of the fluid above the plate (damped with the distance from the plate).
We verify that for

$$\delta = \left(\frac{2\nu}{\omega}\right)^{1/2}$$

, the amplitude is damped to its value at z=0.
3/ Calculate the ratio δ/L.

$$\frac{\delta}{L} = \left[\left(\frac{2\nu}{\omega}\right)\left(\frac{\omega^2}{4U^2}\right)\right]^{1/2} = \left(\frac{\nu}{u\,.\,L}\right)^{1/2} = Re^{-1/2}$$

4/ 4 b - What is the meaning of this report?

$\frac{\delta}{L} = \frac{1}{\sqrt{Re}}$, The effect of viscosity is limited to a layer of relative thickness inversely proportional to the square root of R.

Exercises related to chapter 4

<u>Exercice 1</u>

Are low-speed, small-scale air and water boundary layers really thin? Consider flow at U = 0.305m/s past a flat plate 0.305m long. Compute the boundary layer thickness at the trailing edge for (a)air and(b) water at 293k. $\nu_{air} = 1.5\ 10^{-5} m^2/s$ $\nu_{water} = 1.003\ 10^{-7} m^2/s$

<u>Solution</u>

Part (a)

The trailing-edge Reynolds number thus is:$Re_L = \frac{U\,L}{\nu} = \frac{0.305\times0.305}{0.15\ 10^{-5}} = 6200$

Since this is less than 10^6, the flow is presumed laminar, and since it is greater than 2500, the boundary layer is reasonably thin. the predicted laminar thickness is:

$$\frac{\delta}{x} = \frac{5}{\sqrt{6200}} = 0.0634$$

At $x = 0.305m \ \rightarrow \qquad \delta = 0.0634 \ \times 0.305 = 0{,}0193m$

Part (b)

The trailing-edge Reynolds number thus is:$Re_L = \frac{U\,L}{\nu} = \frac{0.305 \times 0.305}{1.003\ 10^{-7}} = 92600$

the boundary layer is reasonably thin. the predicted laminar thickness is:

$$\frac{\delta}{x} = \frac{5}{\sqrt{92600}} = 0.0164$$

This again satisfies the laminar and thinness conditions. The boundary layer thickness is

At $x = 0.305m \ \rightarrow \qquad \delta = 0.0164 \ \times 0.305 = 0{,}00508m = 5.08mm$

Exercice 2

A long, thin flat plate is placed parallel to a 0.01524 m/s stream of water at 293k. At what distance x from the leading edge will the boundary layer thickness be 0.0254m?

Property values : $\nu_{water} = 1.003\ 10^{-7} m^2/s$

Solution

Assumptions: Flat-plate flow, with applying in their appropriate ranges.

Approach: Guess laminar flow first. If contradictory, try turbulent flow

1) **try laminar flow**This is impossible, since laminar boundary layer flow only persists up to about 10 6 (or, with special care to avoid disturbances, up to 3 × 10 6). Solution step 2: Try turbulent flow

   Check Rex = (20 ft/s)(5.17 ft)/(1.082E-5 ft 2 /s) = 9.6E6 > 10 6 . OK, turbulent flow. Comments: The flow is turbulent, and the inherent ambiguity of the theory is resolved.

Exercises related to chapter 6

Exercise 01:

Oil with an absolute viscosity of 0.101 Pa.s and a density of 0.850 flows through 3000 m of cast iron pipe with a diameter of 300 mm at a rate of 44.4 l/s.

What is the pressure loss in the pipe?

Solution

$$v = \frac{Q_V}{S} = \frac{Q_V}{\frac{1}{4}.\pi.d^2} = \frac{44{,}4}{0{,}24 \times \pi \times 0{,}3^2} = 0{,}628\, m/s \text{ et } Re = \frac{\rho.d.v}{\mu} = \frac{0{,}850 \times 1000 \times 0{,}3 \times 0{,}628}{0{,}101} = 1585$$

$$v = \frac{Q_v}{\pi\, r^2} = 0.628 m/s \quad R_e = \frac{v\rho\, D}{\mu} = 1585 < 2000$$

Which means that the flow is laminar.

$$\lambda = \frac{64}{Re} = 0.0404 \; and \;\; the\ pressure\ losse\ is\ \lambda \frac{L}{D}\frac{\rho v^2}{2} = 8.14m$$

Exercice 2

Calculate the pressure loss for 305 m of new cast iron pipe, without coating, with an internal diameter of 305 mm, when:

a) water at 15.6 °C flows at 1.525 m/s

b) fuel oil at 15.6 °C flows at the same speed

Solution

When using the Moody diagram, one must first evaluate the relative roughness and then calculate the Reynolds number.

a) Pressure loss for water$\frac{\varepsilon}{D} = \frac{0.244}{305} = 0.0008$

$$R_e = \frac{V\,D}{\nu} = \frac{0.305 \times 1.525}{1.13\ 10^{-6}} = 411000 : turbulent\ flow$$

According to the Moody diagram:

$$\frac{\varepsilon}{D} = 0.0008\ and\ Re = 411000 \rightarrow \lambda = 0.0194$$

The pressure loss is :$\Delta H = \lambda \frac{L}{D} \frac{\rho v^2}{2} = 0.0194.\frac{305}{0.305}.\frac{1.525^3}{2\ .9.81} = 2.3m$

b)Pressure loss for fuel

Pressure loss for water$\frac{\varepsilon}{D} = \frac{0.244}{305} = 0.0008$

$$R_e = \frac{V\,D}{\nu} = \frac{0.305 \times 1.525}{4.41\ 10^{-6}} = 105000 : turbulent\ flow$$

According to the Moody diagram:

$$\frac{\varepsilon}{D} = 0.0008\ and\ Re = 411000 \rightarrow \lambda = 0.0213$$

The pressure loss is :$\Delta H = \lambda \frac{L}{D} \frac{\rho v^2}{2} = 2.53m$

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